Intro to course

web page https://cs.uwaterloo.ca/~alubiw/CS763.html
Piazza https://piazza.com/uwaterloo.ca/fall2022/cs763/home

Credit:

- 5 assignments (roughly 2 questions each) (50%)
- a project (50%). Pick some topic that interests you and is relevant to the course; explore some aspect of it. You may attempt original research or report on some papers (one paper deeply or a few papers less deeply). You must do a written report and a class presentation. I will suggest possible topics.

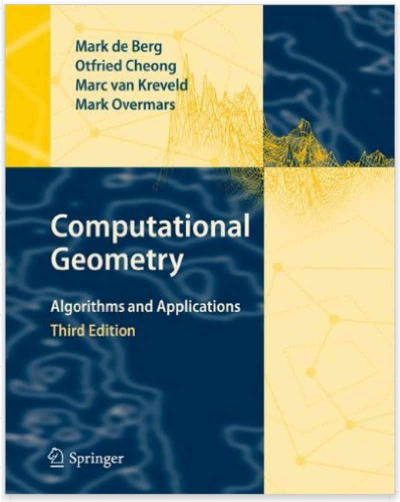
Course Outline

- polygon triangulation
- visibility and guarding
- convex hulls
- linear programming
- Voronoi diagrams and Delaunay triangulations
- surface reconstruction
- arrangements and duality
- geometric data structures, search problems
- motion planning, shortest paths
- curves, trajectories, Fréchet distance

Background: I will assume a background in algorithms and data structures from a decent undergraduate course (e.g. UW's CS 341).

Resources

text:
Computational Geometry: Algorithms and Applications
[CGAA] “the 3 Marks book”



https://ocul-wtl.primo.exlibrisgroup.com/permalink/01OCUL_WTL/156lh75/cdi_proquest_ebookcentral_EBC3062982

lecture notes:
Geometry: Combinatorics & Algorithms
[Zurich notes]

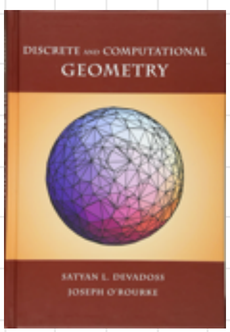
https://geometry.inf.ethz.ch/gca18.pdf

to find papers

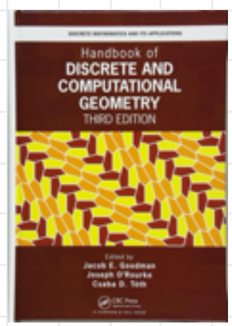
https://scholar.google.com

other good books

Discrete and Computational Geometry
[Devadoss-O’Rourke]



Handbook of Discrete and Computational Geometry
[Handbook]



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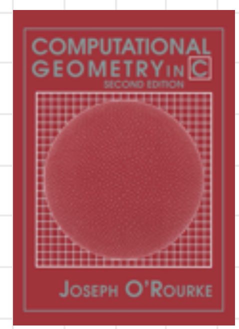
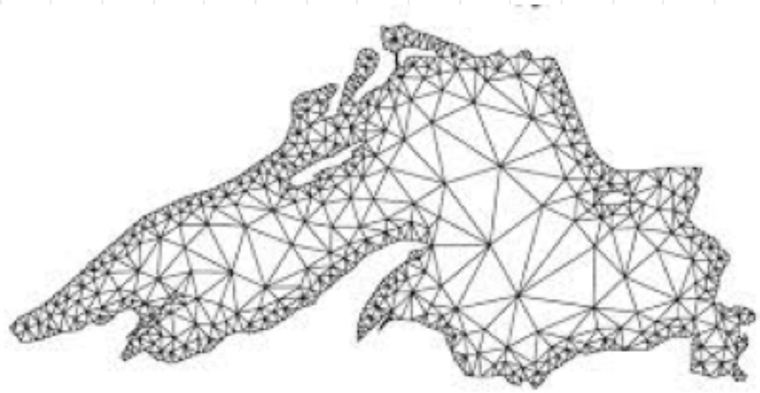
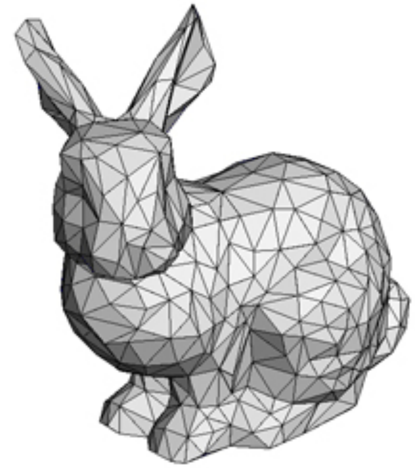
Computational Geometry in C
[O’Rourke]



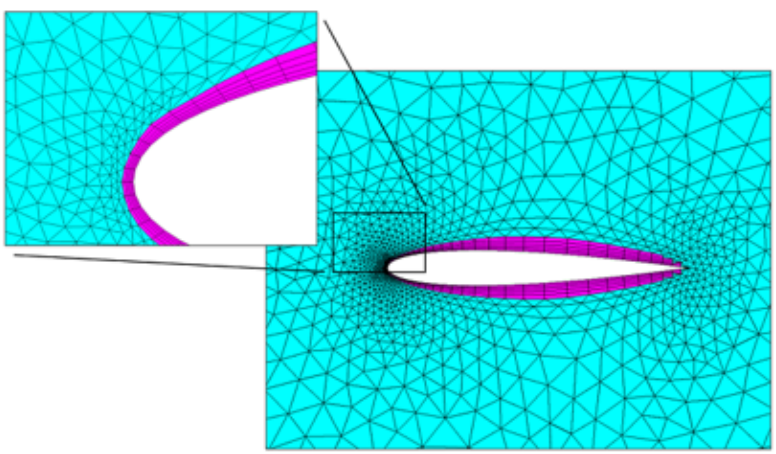
Triangulate a polygon/point set/surface



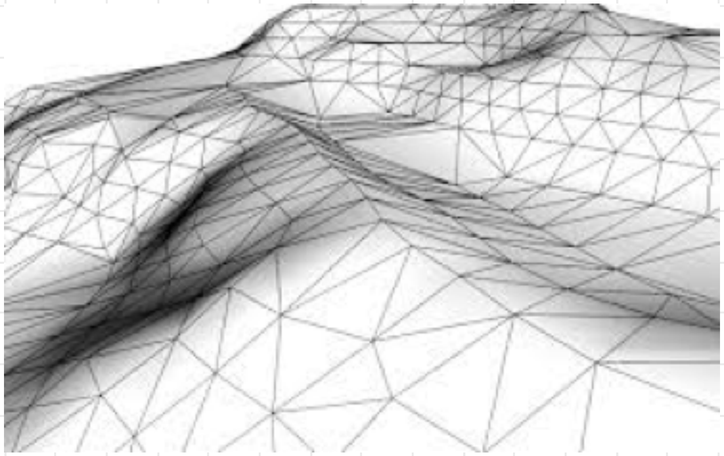
polygon



polyhedron



polygonal region



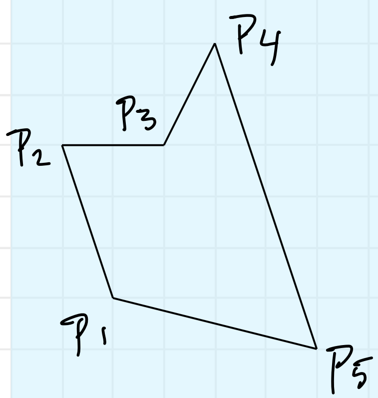
terrain

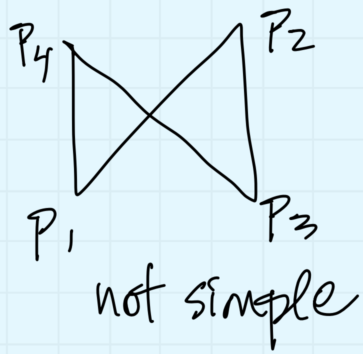
Polygon Triangulation

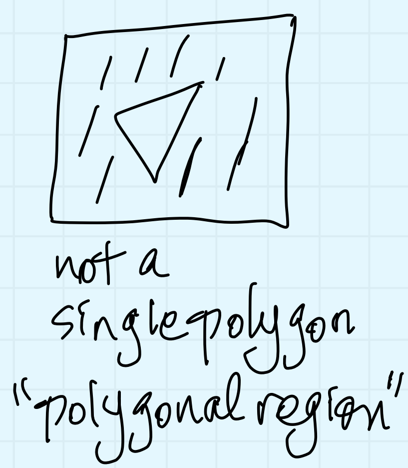
Definition. A ***polygon*** is specified by a sequence of points in the plane, $p_1, p_2, \ldots, p_n$ called ***vertices***. The ***edges*** are the line segments $e_i = p_i p_{i+1}$

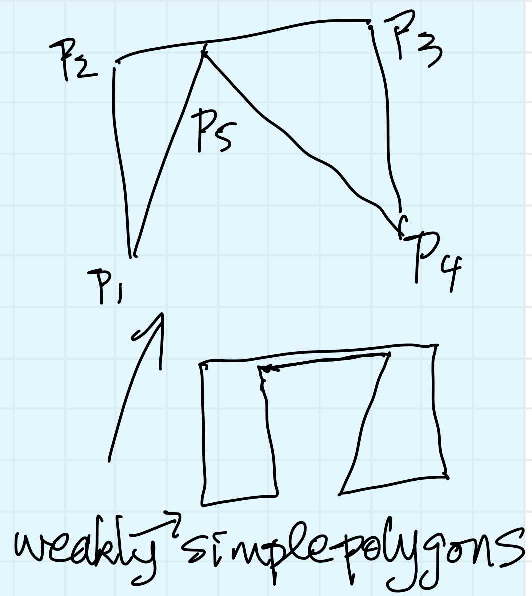
We assume ***simple*** polygons — two edges intersect only at a common vertex.

Examples









How do we test if a polygon is simple? Plane-sweep $O(n \log n)$.

. . . weakly simple?

Recognizing weakly simple polygons
HA Akitaya, G Aloupis, J Erickson, CD Tóth - Discrete & Computational ..., 2017 - Springer

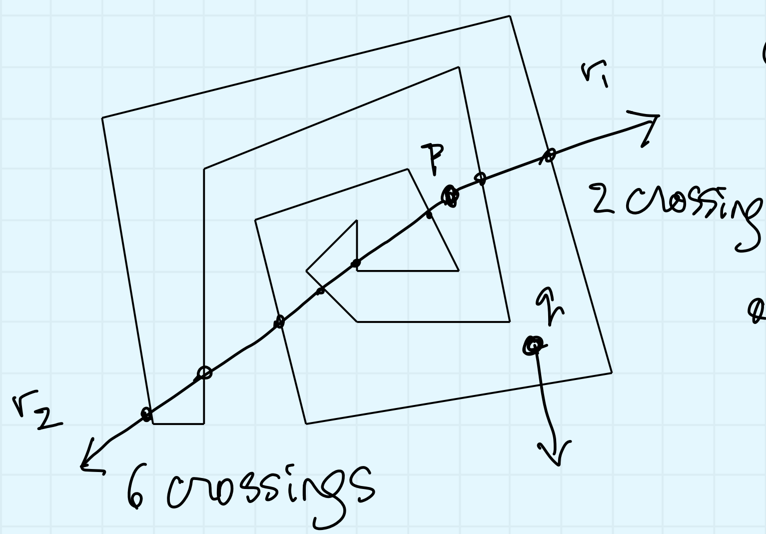
Jordan Curve Theorem

A simple polygon divides the plane into two regions, the ***inside*** and the ***outside***.

True more generally for simple curves. https://en.wikipedia.org/wiki/Jordan_curve_theorem

Elementary proof for polygons — Courant and Robbins, 1941
https://en.wikipedia.org/wiki/What_Is_Mathematics%3F

How to test if a point is inside/outside a polygon:



Construct a ray r from p
Count # crossings.
odd # crossing $\Leftrightarrow$ p is inside

Motivation for Decomposing Polygons

Most algorithms on polygons work better on small/nice polygons — triangles or convex pieces.

Note: 3D is more useful than 2D but we often work with *surfaces* in 3D and these are stored as a collection of polygons.

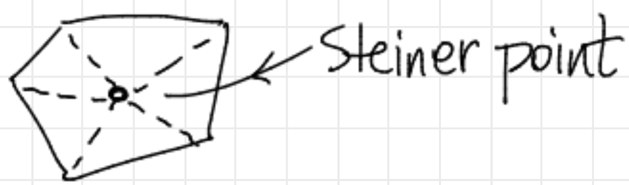
Types of Decompositions

- partition — express polygon as union of disjoint subpolygons
- covering — express polygon as union of subpolygons
- Boolean combination — express polygon as Boolean combination (union, intersection, minus, etc.) of subpolygons.

Steiner points

Sometimes we require the subpolygon vertices to be vertices of the original.
Otherwise the new vertices are called ***Steiner points***.
Examples:

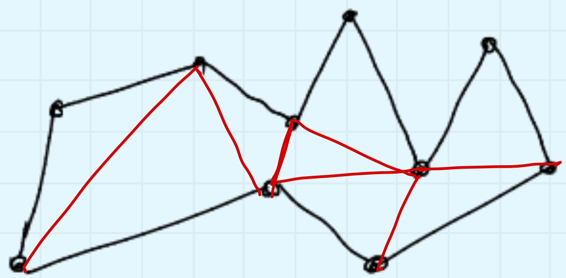
versus



Triangulating Polygons

Partition a polygon into triangles without Steiner points. Each triangle edge will be a ***chord*** — a line segment inside the polygon joining two vertices.

Example



Theorem [Lennes 1911] Any polygon P can be triangulated.

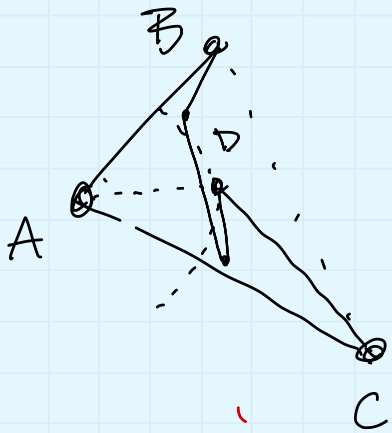
Proof. By induction on # vertices. Basis $n=3$ — have one △

Enough to find one chord — line segment inside P joining two vertices.

A chord divides P into 2 smaller pieces — by induction can triangulate them.

Theorem [Lennes 1911] Any polygon can be triangulated.

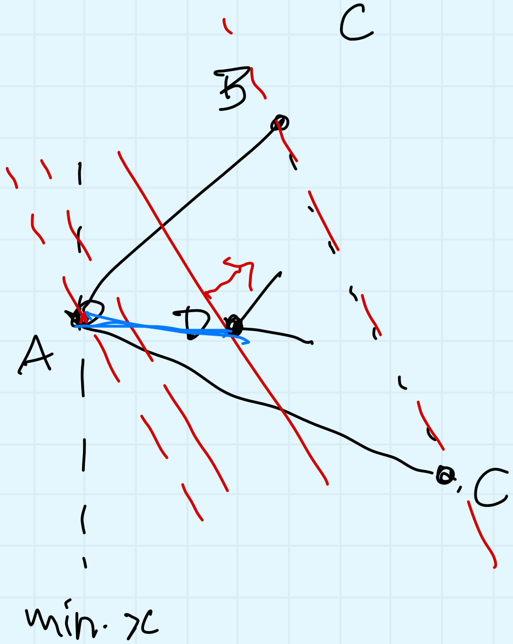
Proof. Take a vertex with angle < 180° ("convex vertex")
e.g. min x-coord (and min y-coord in case of ties).



Hope: BC is a chord.

If not find a vertex D inside △ABC
s.t. AD is a chord.

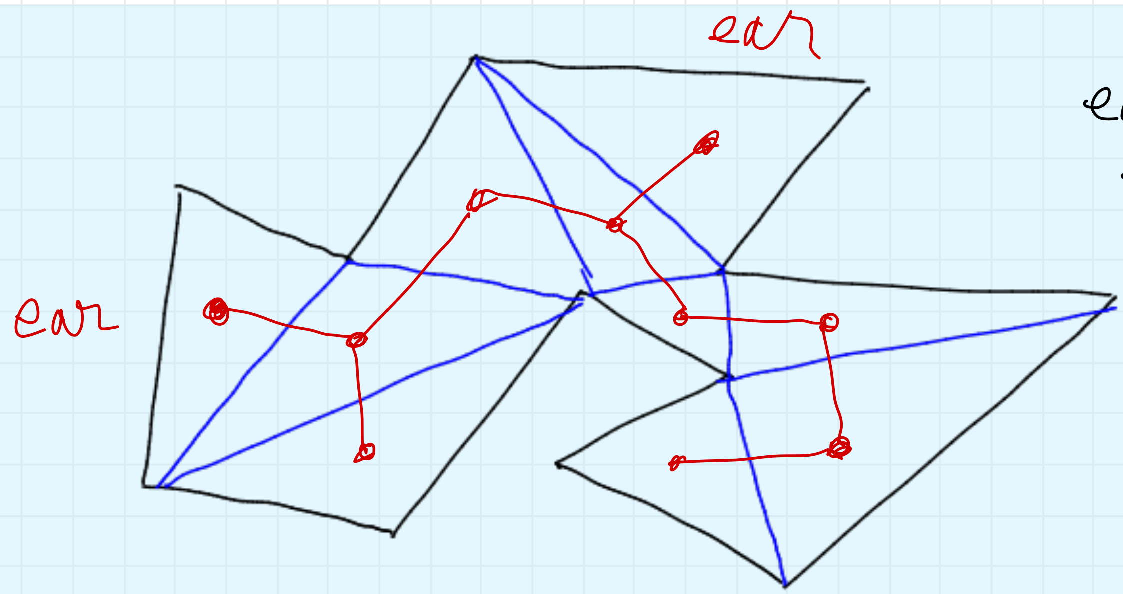
Careful: D closest to A fails!



Sweep line parallel to BC
from A towards BC.
If we get to BC then BC is a chord.
Otherwise we hit a vertex D
Then AD is a chord. ☒

Some properties of polygon triangulations

- number of triangles is $n-2$ (# chords is $n-3$)
- every polygon has (at least) two disjoint ***ears*** = triangle formed by 2 incident polygon edges + 1 chord.
- triangles form a tree

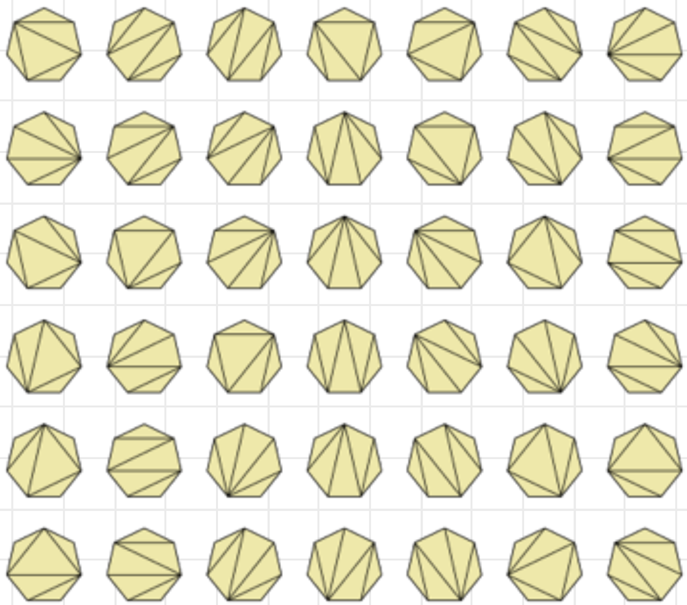


ears are leaves of the dual tree

degrees of tree vertices: 1, 2, or 3

Exercise. Prove that the sum of the interior angles of any polygon is $\pi(n-2)$

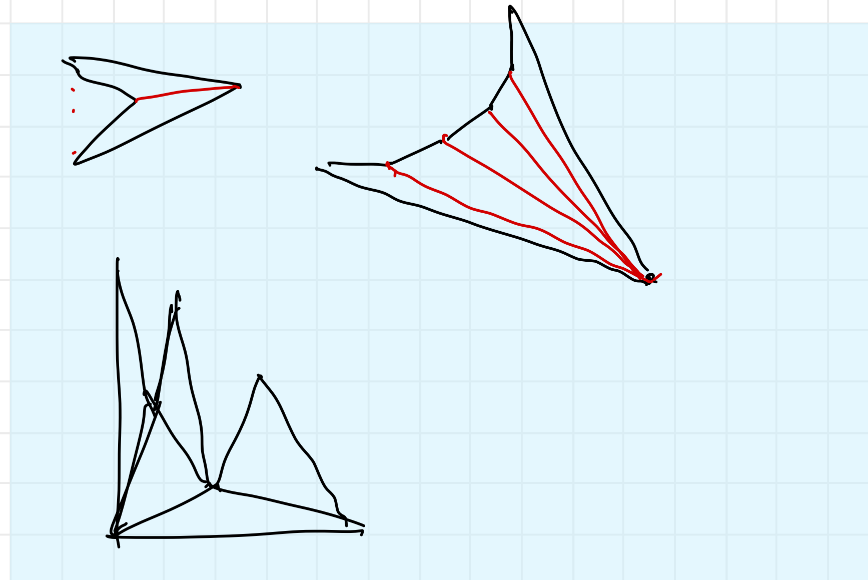
The number of triangulations of a polygon.



The 42 possible triangulations for a convex heptagon

W https://en.wikipedia.org/wiki/Polygon_triangulation

Some polygons have a unique triangulation.



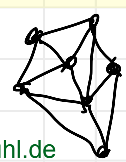
Fact: The number of triangulations of an n-vertex convex polygon is the Catalan number C_{n-2}

Problem: Give a polynomial-time algorithm to compute the number of triangulations of a simple polygon.

for point sets

Peeling and Nibbling the Cactus: Subexponential-Time Algorithms for Counting Triangulations and Related Problems

D Marx, T Miltzow - 32nd International Symposium on …, 2016 - drops.dagstuhl.de

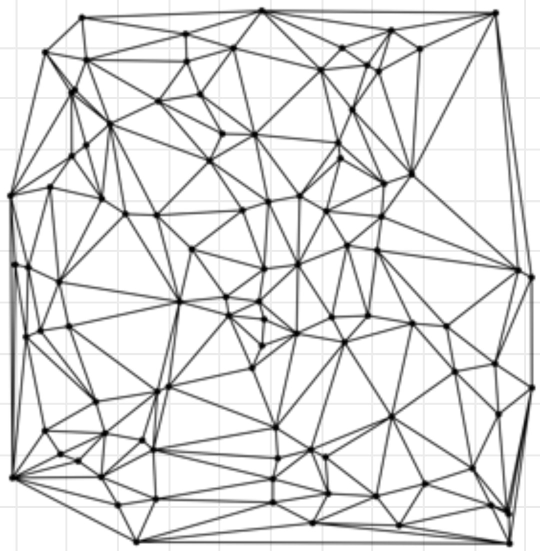


for polygonal regions

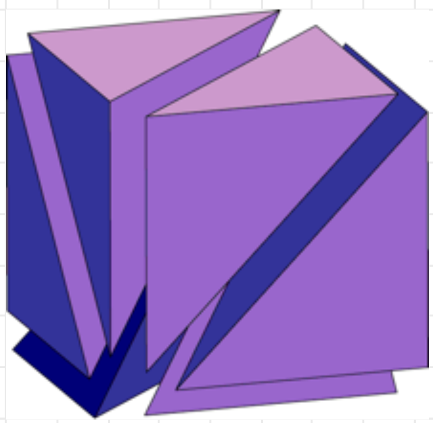
Counting Polygon Triangulations is Hard

D Eppstein

Later on we will talk about triangulating point sets in the plane



Also about “triangulating” polyhedra in 3D



Exercise: cut a cube into min. number of tetrahedra.

Algorithms to triangulate a polygon

1. obvious method (find a chord, following the proof) takes O(n^4)

   can be improved to O(n^2) by cutting off ears

   Exercise: figure out the details for this

2. O(n log n) algorithm next day

3. O(n log* n) randomized algorithm of Seidel (faster than O(n log n))

4. Optimal algorithm to triangulate a polygon O(n)

Bernard Chazelle 1991

Triangulating a simple polygon in linear time

B Chazelle - Discrete & Computational Geometry, 1991 - Springer

Abstract. We give a deterministic algorithm for triangulating a simple polygon in linear time. The basic strategy is to build a coarse approximation of a triangulation in a bottom-up phase and then use the information computed along the way to refine the triangulation ...

But so complicated that there's no implementation!

Linear-time polygon triangulation has intriguing consequences. For example, one cannot check in linear time whether a list of segments ab, cd, ef, gh, etc, is free of intersections, but if the list is of the form ab, bc, cd, de, etc, then miraculously one can. Segueing into my favorite open problem in plane geometry, can the self-intersections of a polygonal curve be computed in linear time? I know the answer (it's yes) but not the proof.

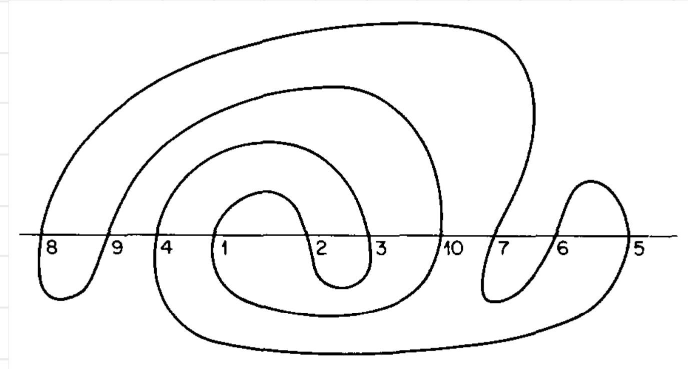
https://www.cs.princeton.edu/~chazelle/linernotes.html

The power of having a simple polygon

[Simplified linear-time Jordan sorting and polygon clipping]

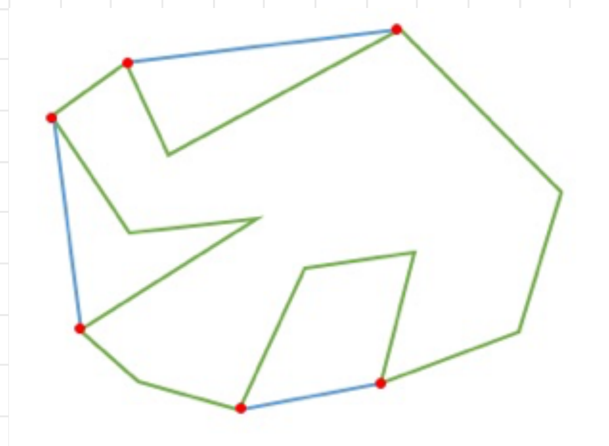
KY Fung, TM Nicholl, RE Tarjan, CJ Van Wyk - Information Processing …, 1990 - Elsevier

Given the intersection points of a Jordan curve with the x-axis in the order in which they occur along the curve, the Jordan sorting problem is to sort them into the order in which they occur along the x-axis. This problem arises in clipping a simple polygon against a rectangle (a "window") and in efficient algorithms for triangulating a simple polygon. Hoffman, Mehlhorn, Rosenstiehl, and Tarjan proposed an algorithm that solves the Jordan sorting problem in time that is linear in the number of intersection points, but their algorithm requires …



[ON-LINE CONSTRUCTION OF THE CONVEX HULL OF A SIMPLE POLYLINE]

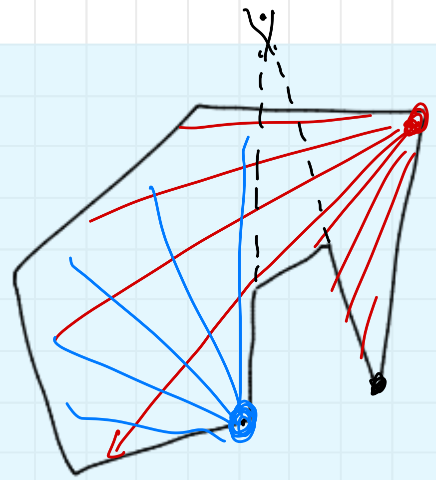
AA Melkman - Information Processing Letters, 1987 - ime.usp.br



Art Gallery Theorem (an application of triangulations)

Regard a polygon as a floorplan of an art gallery, edges = walls.
How many guards are needed to watch the whole gallery?

Example



2 guards work

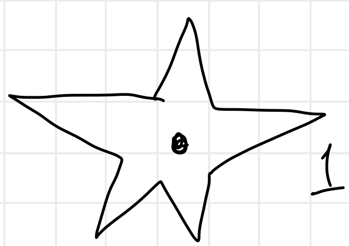
1 guard is not enough.

Problem posed by Victor Klee 1973, bound proved by Chvatal 1975, simple proof by Fisk 1978.

$\lfloor \frac{n}{3} \rfloor$ vertex guards

Theorem. For an *n* vertex polygon floor(*n*/3) guards always suffice, and for some *n*-vertex polygons, floor(*n*/3) guards are necessary.

^ situated anywhere.



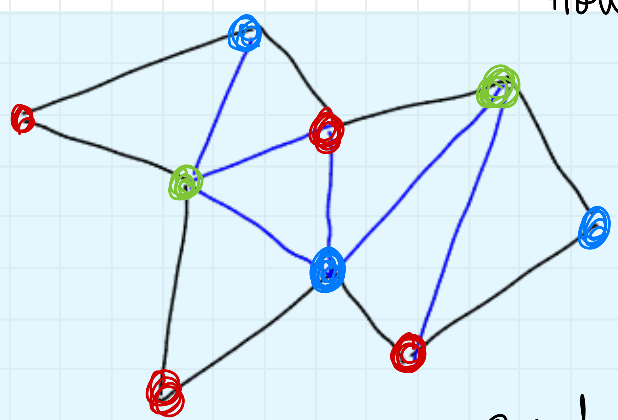
1 guard suffices but not on a vertex.

Theorem. For an *n* vertex polygon floor(*n*/3) guards always suffice, and for some *n*-vertex polygons, floor(*n*/3) guards are necessary.

Proof [Fisk] Triangulate the polygon.

Colour the vertices red, green, blue s.t. every triangle has every colour.

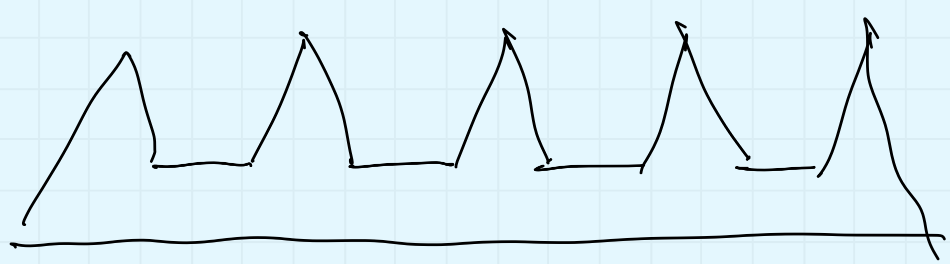
How to colour:



cut off an ear — by induction colour the smaller piece
Then add ear back

The smallest colour class has $\le \lfloor \frac{n}{3} \rfloor$ vertices and guards the polygon (because it guards every △)

When are floor(*n* /3) guards necessary?



n = 15 vertices
need 5 guards
(one per "tooth")

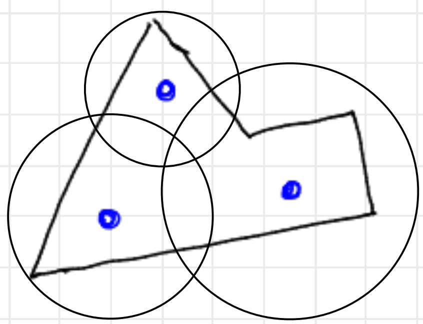
There are many further results on guarding.

Exercise: If the polygon is orthogonal, make a conjecture about the number of guards that are always sufficient and sometimes necessary.

Art Gallery Theorems and Algorithms, Joseph O'Rourke, 1987

http://cs.smith.edu/~jorourke/books/ArtGalleryTheorems/

Guards may have limited visibility — sensor networks



Efficient sensor placement for surveillance problems

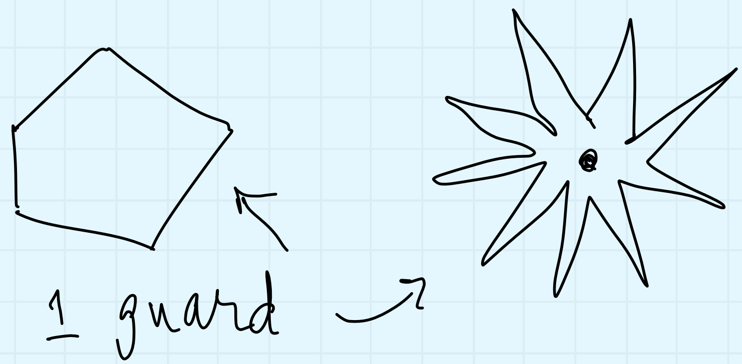
PK Agarwal, E Ezra, SK Ganjugunte - International Conference on …, 2009 - Springer

Algorithms for the Art Gallery Problem

Is there an algorithm to find the minimum number of guards for a given polygon?
(Above results were about the worst-case number of guards for an n-gon.)

Guards need not be on the boundary.

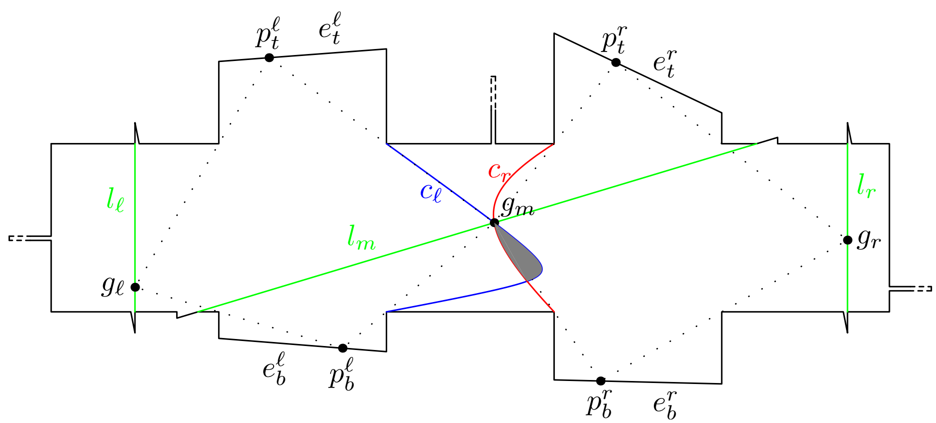
Example:



The problem is NP-hard. Is the decision problem in NP?

Geometric problems involve issues of real numbers!
What is our model of computing? How do we deal with imprecise points?

Guards might need to be at irrational points! (to get min. number of guards)



Irrational Guards are Sometimes Needed
M Abrahamsen, A Adamaszek… - … Geometry (SoCG 2017), 2017 - drops.dagstuhl.de

The Art Gallery Problem is hard for existential theory of the reals $\exists\mathbb{R}$

The art gallery problem is∃ ℝ-complete
M Abrahamsen, A Adamaszek, T Miltzow - Proceedings of the 50th …, 2018 - dl.acm.org

$$P \subseteq NP \subseteq \exists\mathbb{R} \subseteq PSPACE$$

A Practical Algorithm with Performance Guarantees for the Art~ Gallery Problem
S Hengeveld, T Miltzow - arXiv preprint arXiv:2007.06920, 2020 - arxiv.org

Summary

- polygon, triangulation, art gallery problem
- two proofs: polygons can be triangulated; n/3 art gallery guards
- dangers of real numbers in geometric problems
- algorithms! possible, impossible, un-implementable

References

- [CGAA] Section 3.1
- [Zurich notes] Chapter 3
- [O'Rourke] 1.1, 1.2
- [Devadoss-O'Rourke] 1.1 - 1.3