-ncfactor.lib-
Factoring in noncommutative polynomial rings

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Abstract

We present `ncfactor.lib`, a SINGULAR [1] library dealing with the factorization problem in the polynomial first Weyl algebra utilizing its $\mathbb{Z}$-graded structure. The same ideas are applicable to the polynomial first shift algebra, as well as to the polynomial first q-Weyl algebra. Algorithms for all three algebras are implemented in `ncfactor.lib`.

We are going to present the general ideas and provide some examples demonstrating the advantages of our approach. The leitmotif is to reduce it to commutative problems as far as one can.

`ncfactor.lib` is distributed with SINGULAR since version 3-1-3; we developed the techniques further and there was a crucial change made in version 3-1-6, increasing the performance and the versatility of the results significantly.

Problem Background

The Weyl algebra is an algebraic abstraction of the differential operator acting on a variable x. One can introduce the following linear operators on $\mathbb{K}[x]$:

$$\mathcal{X} : \mathbb{K}[x] \to \mathbb{K}[x], f \mapsto x \cdot f, \qquad \mathcal{D} : \mathbb{K}[x] \to \mathbb{K}[x], f \mapsto \frac{df}{dx},$$

where $\frac{df}{dx}$ denotes the formal differentiation on $\mathbb{K}[x]$. Furthermore consider the composition $\mathcal{D} \circ \mathcal{X} := \mathcal{D}\mathcal{X}$. The Leibniz rule grants for all $f \in \mathbb{K}[x]$

$$\mathcal{D}\mathcal{X}f = x\frac{d}{dx}f + f = (\mathcal{X}\mathcal{D} + 1)f.$$

With this formula, we get a commutation rule for $\mathcal{D}$ and $\mathcal{X}$, namely $\mathcal{D}\mathcal{X} = \mathcal{X}\mathcal{D} + 1$.

Hence, we can consider differential operators as members of a ring and perform computations like factorization on them. Factorizing a differential operator provides the benefit of solving the associated differential equation by splitting it into smaller pieces.

The q-Weyl algebra on the other hand is an algebraic abstraction of the q-differentiation.

Details

Homogeneous polynomials in A_1

The main idea relies on the following fact:

Lemma 1. *We have $Q_1^{(0)} = \mathbb{K}(q)[\theta]$. Moreover, $Q_1^{(k)}$ is an $Q_1^{(0)}$-module generated by the element x^{-k}, if $k < 0$, or by ∂^k, if $k > 0$.*

Furthermore, it holds that

- $f(\theta)x^n = x^n f(q^n\theta + \frac{q^n-1}{q-1})$ and $f(\theta)\partial^n = \partial^n f\left(\frac{1}{q}\left(\frac{\theta-1}{q^{n-1}} - \frac{q^{-n+2}-q}{1-q}\right)\right)$ in Q_1,
- $f(\theta)\partial^k = \partial^k f(\theta - k)$ and $f(\theta)x^k = x^k f(\theta + k)$ in A_1,

for $f(\theta) \in \mathbb{K}(q)[\theta]$ and $n \in \mathbb{N}$. Thus we only have to deal with the factorization of $f(\theta)$ and get the other possible factorizations by permutation with x resp. ∂.

What happens if we map back from $K(q)[\theta]$ to Q_1 resp. A_1? What polynomials will appear to be reducible, that were irreducible before? The next Lemma provides an unexpected answer.

Lemma 2. *The polynomials θ and $\theta + \frac{1}{q}$ are the only irreducible monic elements in $\mathbb{K}[\theta]$ that are reducible in Q_1. For A_1, the polynomials θ and $\theta + 1$ are the only irreducible monic elements in $\mathbb{K}[\theta]$ that are reducible in A_1.*

Corollary 3. *The factorization problem in Q_1 and A_1 can be completely reduced to the factorization problem in a univariate commutative polynomial ring over $\mathbb{K}(q)$.*

Arbitrary polynomials in A_1

Remark 4. *The ideas for arbitrary polynomials are also applicable to Q_1, but not implemented yet. Therefore, we will restrict ourselves to A_1 here.*

Let

$$h = \sum_{i=\hat{n}}^{n} h_i \in A_1, \hat{n} < n \in \mathbb{Z},$$

$h_i \in A_1^{(i)}$ for $i \in \{\hat{n}, \ldots, n\}$, be the polynomial we want to factorize and let us assume that it possesses a nontrivial factorization of at least two factors. Let us denote those – not necessary irreducible – factors by

$$h = \sum_{i=\hat{n}}^{n} h_i := (p_{n_1} + \ldots + p_{n_k})(q_{m_1} + \ldots + q_{m_l}) = \sum_{i=\hat{n}}^{n} \sum_{\substack{j_1+j_2=i \\ (j_1,j_2)\in \underline{k}\times\underline{l}}} p_{j_1}q_{j_2}, \tag{1}$$

where $k, l \in \mathbb{N}$, $n_1 > n_2 > \ldots > n_k$ and $m_1 > m_2 > \ldots > m_l \in \mathbb{Z}$, $p_{n_i} \in A_1^{(n_i)}$ for all $i \in \underline{k}$, $q_{m_j} \in A_1^{(m_j)}$ for all $j \in \underline{l}$.

From the equation (1) we see directly that $h_{\hat{n}} = p_{n_k}q_{m_l}$ and $h_n = p_{n_1}q_{m_1}$.

As homogeneous factorization is a solved task, we can consider $p_{n_1}, p_{n_k}, q_{m_1}$ and q_{m_l} to be known to us and not to be zero.

It remains to solve for the other p_{n_i} and q_{m_j} for $(i,j) \in \underline{k} \times \underline{l}$. For that, we will use Lemma 1 and define for all $i \in \underline{k}$ the polynomial $\tilde{p}_{n_i}$ by $\tilde{p}_{n_i}\partial^{n_i} = p_{n_i}$, if $n_i \geq 0$ and $\tilde{p}_{n_i}x^{-n_i} = p_{n_i}$, if $n_i < 0$, where the $\tilde{p}_{n_i} \in A_1^{(0)}$. In the same way we define $\tilde{q}_{m_i}$ for all $i \in \underline{l}$ and $\tilde{h}_i$ for $i \in \{\hat{n}, \ldots, n\}$.

Theorem 5. *The correct solution for the $\tilde{p}_{n_2}, \ldots, \tilde{p}_{n_{k-1}}, \tilde{q}_{m_2}, \ldots, \tilde{q}_{m_{l-1}}$ is a polynomial solution of degree at most $\min\{\deg_x(h), \deg_\partial(h)\}$ in θ of the following set of equations with the known parameters $\tilde{h}_{\hat{n}}, \ldots, \tilde{h}_n$ and $\tilde{p}_{n_1}, \tilde{p}_{n_k}, \tilde{q}_{m_1}, \tilde{q}_{m_l}$:*

$$\left\{ \sum_{\substack{j_1,j_2\in\underline{k}\times\underline{l} \\ n_{j_1}+m_{j_2}=i}} \tilde{p}_{n_{j_1}}\tilde{q}_{m_{j_2}} \circ (\theta + n_{j_1})\gamma_{n_{j_1},m_{j_2}} = \tilde{h}_i \mid \hat{n} = n_k + m_l < i < n_1 + m_1 = n \right\},$$

where γ_{ij} are certain Pochhammer symbols.

Lemma 6. *We can simplify the equations in Theorem 5, such that we only have to consider the $\tilde{q}_{m_j}, j \in \underline{l}$ as indeterminates and obtain them solving a system of difference equations in $\mathbb{K}[\theta]$ or a nonlinear system of equations, alternatively.*

Comparison to other computer algebra systems

We compared our implementation in the computer algebra system SINGULAR to MAPLE ([2, 3], version 16, procedure: `DFactor` in the `DETools` library) and REDUCE ([4], version 3.8, procedure: `nc_factorize[_all]` in the library `NCPOLY`). MAPLE aims at factorization over the rational Weyl algebra, while REDUCE and our implementation are factorizing elements over the polynomial Weyl algebra.

For generic polynomials the computation times for the factorization were similar in all implementations. But for $[-1,1]$-homogeneous polynomials our method often delivers factorizations of a given polynomial quite fast while the MAPLE and REDUCE calculations were cancelled after running for more than nine hours (indicated by – NT – in the table).

Considering the obtained factorizations, `ncfactor.lib` does not utilize field extensions of any kind. It appears that the solutions are therefore often more "neat". Consider for example the third polynomial in the table below. Singular returns $(x^5\partial^5 + 6) \cdot (x^5\partial^5 + x^3\partial^3 + 4)$ and $(x^5\partial^5 + x^3\partial^3 + 4) \cdot (x^5\partial^5 + 6)$ as possible factorizations, whereas the output of MAPLE is a 102KB file starting like this: $x^{10}d+1/2*x^9(3+RootOf(\_Z^3+(-3-2RootOf(\_Z^4+(-69-3RootOf(\_Z^5+15\_Z^4+86\_Z^3+237\_Z^2+321\_Z+184))\_Z^3+[\ldots]$

SINGULAR	MAPLE	REDUCE
$(x\partial^2 + x\partial + 1 + (x\partial + 5) \cdot x) \cdot (((x\partial)^2 + 1)\partial + x\partial + 3 + (x\partial + 7) \cdot x)$:		
12.05s; #fcts.: 1	0.63s; # fcts.: 1	0.78s; #fcts.: 1
$(x\partial^2 + (x\partial)^5 + x) \cdot ((x\partial + 1)\partial - (x\partial - 1)^5 + x)$:		
3.72s; #fcts.: 1	6.62s; #fcts.: 1	– NT –
$(x^5\partial^5 + 6) \cdot (x^5\partial^5 + x^3\partial^3 + 4)$:		
0.12s; #fcts.: 2	31.58s; #fcts.: 1	– NT –
$(x^{10}\partial^{10} + 5x\partial + 7) \cdot x^2 \cdot (x^{11}\partial^{11} + 3x^7\partial^7 + x\partial + 4)$:		
0.34s; #fcts.: 12	– NT –	– NT –
$(5x^{10}\partial^{10} + 7x^9\partial^9 + 8x^8\partial^8 + 9x^7\partial^7 + 6x^6\partial^6 + 5 + 5x^5\partial^5 + 8x^4\partial^4 + 5x^3\partial^3 + 9x^2\partial^2 + 9x\partial + 6)\partial^{20}$:		
0.96s; #fcts.: 21	– NT –	– NT –
$(x^4 - 1)x\partial^2 + (1 + 7x^4)\partial + 8x^3$ ([5], p. 200):		
0.75s; #fcts.: 12	0.75s ;#fcts.: 1	3.27s;#fcts.: 60 *

* Some of the factors were reducible.

Conclusion

`ncfactor.lib` broadens the current range of polynomials that can be factorized using a computer algebra system. Furthermore, we do not leave the ground field in order to obtain our factorizations, which makes the results often more useful in practice.

References

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[3] M. van Hoeij. *Factorization of linear differential operators*. Nijmegen, 1996.

[4] H. Melenk and J. Apel. *REDUCE package NCPOLY: Computation in non-commutative polynomial ideals*. Konrad-Zuse-Zentrum Berlin (ZIB), 1994.

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Notations and definitions: $\mathbb{K}$ denotes an arbitrary field of characteristic zero; θ denotes the so called Euler operator (in the first (q-)Weyl algebra: $\theta = x\partial$); Q_1 denotes the first q-Weyl algebra over $\mathbb{K}(q)$ (either $q \in \mathbb{K}$, or an algebraic, or a transcendent element) with the indeterminates x, ∂ satisfying $x_i\partial_j - q\partial_j x_i = \delta_{ij}$ (Kronecker Delta). For $q = 1$, we have simply the first Weyl algebra, denoted by A_1. Furthermore, the mentioned grading on Q_1 resp. A_1 is induced by the weight vector $[-1, 1]$. $A_1^{(n)} := \left\{\sum_{j-i=n} r_{i,j}x^i\partial^j | i, j \in \mathbb{N}_0, r_{i,j} \in \mathbb{K}\right\}$ denotes the set of homogeneous polynomials in A_1 of degree $n \in \mathbb{Z}$; By $\mathbb{S}$ we mean the shift algebra; for $n \in \mathbb{N}$, we denote the set $\{1, \ldots, n\}$ by $\underline{n}$.