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Regularized robust optimization: the optimal portfolio execution case

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Abstract An uncertainty set is a crucial component in robust optimization. Unfortunately, it is often unclear how to specify it precisely. Thus it is important to study sensitivity of the robust solution to variations in the uncertainty set, and to develop a method which improves stability of the robust solution. In this paper, to address these issues, we focus on uncertainty in the price impact parameters in an optimal portfolio execution problem. We first illustrate that a small variation in the uncertainty set may result in a large change in the robust solution. We then propose a *regularized robust optimization* formulation which yields a solution with a better stability property than the classical robust solution. In this approach, the uncertainty set is regularized through a *regularization constraint*, defined by a linear matrix inequality using the Hessian of the objective function and a *regularization parameter*. The regularized

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robust solution is then more stable with respect to variation in the uncertainty set specification, in addition to being more robust to estimation errors in the price impact parameters. The regularized robust optimal execution strategy can be computed by an efficient method based on convex optimization. Improvement in the stability of the robust solution is analyzed. We also study implications of the regularization on the optimal execution strategy and its corresponding execution cost. Through the regularization parameter, one can adjust the level of conservatism of the robust solution.

Keywords Robust optimization · Estimation errors · Portfolio optimization · Price impact

1 Introduction

Uncertainty is inevitable in any real world decision making problem. An optimization problem formulation often relies on model parameters which must be estimated. This presents challenges in the precise notion of optimality and computation of an optimal decision. Several approaches to account for data uncertainty in optimization problems have been proposed in the literature. In particular, *robust optimization* has gained much interest over the last decade, see, e.g., [5, 12]. In robust optimization, parameter uncertainty is modeled deterministically through an *uncertainty set*, which includes all or most possible parameter values. The approach then offers a solution which has the best worst objective value when parameters belong to the uncertainty set.

The current robust optimization methodology, however, has shortcomings. Firstly, it can be conservative in the sense that a robust solution may have poor objective values for many realizations of the data including the nominal one, see, e.g., [13]. Shrinking the uncertainty set using a scaling factor has been a typical technique to alleviate this issue, see, e.g., [9, 10]. An additional problem, which has not been addressed in the current robust optimization literature, is the potential instability of the robust solution to variation in the uncertainty set. Although a robust solution provides protection in the worst scenario for the input parameters of the nominal optimization problem, it does not necessarily guarantee stability of the robust solution with respect to the uncertainty set.

In this paper, we show that a robust solution can potentially be unstable in the sense that a small variation in the uncertainty set may considerably change the robust solution. To illustrate, we focus on the important problem of optimal portfolio execution, with uncertain price impact parameters. Given a price impact model, this problem yields an optimal execution strategy which minimizes the mean and variance of the cost in executing (selling or buying) orders for blocks of assets within a fixed number of time periods. To focus on the main issues regarding robust optimization solutions, we restrict our attention here to a simple linear price impact model and additive market price dynamics, which have been used in one of the most influential literature on optimal portfolio execution [1]. This model has been the building block in several literature on this problem, see, e.g., [37, 47], and the references therein.

Assuming a deterministic strategy, this problem is then reduced to a quadratic programming problem. Unfortunately, estimating the price impact parameters is a challenging task, see, e.g., [54]. Furthermore, it has been shown in [42] that, the optimal execution strategy and the efficient frontier can be sensitive to these estimation errors and may fluctuate substantially. Hence, it is necessary to take estimation errors in the price impact parameters into account when seeking for an optimal execution strategy. Here, we consider a robust optimization technique to handle uncertainty in price impact.

We first use simulation to illustrate potential instability of the classical robust optimization method for the discussed optimal portfolio execution problem with respect to the uncertainty set for price impact parameters. Specifically we show that, for an interval uncertainty set, sensitivity of the robust solution and the robust efficient frontier to perturbations in the boundaries of the uncertainty set can be larger than sensitivity of the nominal solution and the nominal efficient frontier to changes in the nominal price impact parameters. Next we show that, for a convex and compact uncertainty set and convex set of feasible execution strategies, a robust optimal execution strategy uniquely exists, when the Hessian of the objective function is positive definite for every realization of price impact parameters in the uncertainty set. Under this assumption, the unique robust solution can be computed via solving a convex programming problem which yields a worst case realization of the price impact parameters, and the optimal Lagrange multipliers. These values are then used to determine a robust optimal execution strategy.

To improve stability of the robust optimization, we propose the following *regularized robust optimization* approach for the optimal portfolio execution problem, we consider. Given any convex compact uncertainty set, a *regularization constraint* is included to construct a *regularized uncertainty set*. This regularized uncertainty set is then used in the minimax formulation to yield a regularized robust solution. For the optimal portfolio execution problem with uncertain parameters in a linear price impact model, the regularization constraint is a lower bound constraint on the minimum eigenvalue of the Hessian of the objective function. We refer to this fixed lower bound as the *regularization parameter*. The regularization constraint using the eigenvalue function retains convexity of a convex uncertainty set. Varying eigenvalues of some design matrix to enhance stability properties is fairly common in engineering problems, see, e.g., [40].

The intuition behind the proposed regularization constraint comes from the following mathematical result in [42]: variation in the solution can be large, as price impact parameters are perturbed, when the minimum eigenvalue of the Hessian of the objective function corresponding to the pair of price impact parameters is small. By imposing the regularization constraint, we prevent potential instability of the robust solution by excluding elements, which may result in unstable solutions, from the uncertainty set.

We make two main contributions in this paper. Firstly, we study sensitivity of the classical robust optimization to changes in the uncertainty set. Secondly, we propose a regularized robust optimization approach for an optimal portfolio execution problem with uncertain price impact parameters. The regularized robust solution is unique and can be obtained by an efficient method based on convex optimization for a positive regularization parameter. We illustrate that including the regularization constraint in the uncertainty set improves stability of the robust solution. We formally show that the change in the regularized robust optimal execution strategy is bounded above by the change in the worst case price impact parameters over the regularized uncertainty set. In addition, the change in the regularized robust solution converges to zero when the variation in the uncertainty set approaches zero. We then investigate some implications of the regularization on the regularized robust solution and its robust objective function value.

Our presentation is organized as follows. A mathematical formulation of the optimal portfolio execution problem, that we consider, is presented in Sect. 2. The classical robust optimization approach is described in Sect. 3, where we also discuss potential instability of the robust solution to variation in the uncertainty set. Derivation of the robust solution under the assumption that the Hessian of the objective function is positive definite over the uncertainty set is presented in Sect. 4. We propose the regularized robust optimization approach for the optimal portfolio execution problem in Sect. 5. Stability of the approach is discussed in Sect. 6. Several implications of regularization on the regularized robust solution and its objective function value are addressed in Sect. 7. Concluding remarks are given in Sect. 8.

2 Optimal portfolio execution

To reduce price impact of liquidating large blocks of assets over a fixed time interval, an execution strategy typically breaks the holdings into smaller trades and executes them gradually over the trading horizon. Without loss of generality, assume that a trader plans to liquidate his holdings in m assets during N periods in the time horizon T, $t_0 = 0 < t_1 < \cdots < t_N = T$, where $\tau \overset{\text{def}}{=} t_k - t_{k-1} = \frac{T}{N}$ for $k = 1, 2, \ldots, N$. The investor's position at time t_k is denoted by the m-vector $x_k = (x_{1k}, x_{2k}, \ldots, x_{mk})^T$, where x_{ik} is the investor's holding in the ith asset at period k. The investor's initial position is $x_0 = \bar{S}$, and his final position x_N equals 0, which guarantees complete liquidation by time T. The difference between the positions of two successive periods $k-1$ and k is denoted by an m-vector $n_k = x_{k-1} - x_k$ for $k = 1, 2, \ldots, N$. Negative n_{ik} implies that the ith asset is bought between t_{k-1} and t_k. We refer to a sequence $\{x_k\}_{k=0}^N$ satisfying $x_N = 0$ as an *execution strategy*. We build on [1] in forming an optimal portfolio execution strategy.

Let $\tilde{P}_k$ be the execution price of one unit of assets within the time interval $(t_{k-1}, t_k]$, for $k = 1, 2, \ldots, N$. Due to the price volatility, $\tilde{P}_k$ is not deterministic over the execution horizon. Following [1], we assume in this paper that the execution price $\tilde{P}_k$ is given by

$$\tilde{P}_k = P_{k-1} - h\left(\frac{n_k}{\tau}\right), \quad k = 1, 2, \ldots, N, \tag{1}$$

where the market price P_k at time t_k evolves according to the following discrete arithmetic random walk:

$$P_k = P_{k-1} + \tau^{1/2} \Sigma \xi_k - \tau g\left(\frac{n_k}{\tau}\right). \tag{2}$$

Here $\xi_k = (\xi_{1k}, \xi_{2k}, \dots, \xi_{lk})^T$ represents an l-vector of independent standard normals and Σ is an $m \times l$ volatility matrix of the asset prices. The deterministic initial market price, before the trade begins, is denoted by P_0. The functions $g(\cdot)$ and $h(\cdot)$ measure the expected permanent price impact and temporary price impact, respectively. Temporary price impact is the price depression at the moment of trading caused by a trade order. The permanent price impact is the market price change caused by imbalances in supply and demand; this price change persists in the future.

Following [1], we assume the following linear price impact model:

$$g\left(\frac{n_k}{\tau}\right) = G\frac{n_k}{\tau},$$
$$h\left(\frac{n_k}{\tau}\right) = H\frac{n_k}{\tau},$$

where the m-by-m matrices G and H are the permanent and temporary impact matrices, respectively. This model is capable to explain both the permanent and temporary price impacts of large trades, while it is simple enough for mathematical analysis.

The *execution cost* of the trade is defined as $P_0^T \bar{S} - \sum_{k=1}^N n_k^T \tilde{P}_k$. Therefore, an optimal portfolio execution strategy, corresponding to the risk aversion parameter $\mu \geq 0$, can be computed from the following problem:

$$\begin{aligned} \min_{n_1,\dots,n_N} \quad & \mathbf{E}\left(P_0^T \bar{S} - \sum_{k=1}^N n_k^T \tilde{P}_k \right) + \mu \cdot \mathbf{Var}\left(P_0^T \bar{S} - \sum_{k=1}^N n_k^T \tilde{P}_k \right), \\ \text{s.t.} \quad & \sum_{k=1}^N n_k = \bar{S}. \end{aligned} \tag{3}$$

Here, $\mathbf{E}(\cdot)$ and $\mathbf{Var}(\cdot)$ denote the expectation and the variance of a random variable, respectively.

When the execution strategy $\{x_k\}_{k=0}^N$ is assumed to be deterministic as in [1], the expected value and the variance in (3) equal (see, e.g., [42]):

$$\begin{aligned} \mathbf{Var}\left(P_0^T \bar{S} - \sum_{k=1}^N n_k^T \tilde{P}_k \right) &= \tau \sum_{k=1}^N x_k^T C x_k, \\ \mathbf{E}\left(P_0^T \bar{S} - \sum_{k=1}^N n_k^T \tilde{P}_k \right) &= \sum_{k=1}^N x_k^T G(x_{k-1} - x_k) + \frac{1}{\tau} \sum_{k=1}^N (x_k - x_{k-1})^T H (x_k - x_{k-1}). \end{aligned} \tag{4}$$

Here, the $m \times m$ symmetric positive semidefinite matrix $C = \Sigma \Sigma^T$ is the covariance matrix of asset prices. Notice that the variance of the execution cost, under the aforementioned assumptions, does not depend on the impact matrices.

Note that the number of periods N is typically greater than one; otherwise the strategy of liquidating everything in the first period will be the only feasible (whence optimal) execution strategy. The variance of the execution cost corresponding to this strategy equals zero.

Under these assumptions, the (nominal) optimal portfolio execution problem (3) can be formulated as the following quadratic programming problem (see, e.g., [42]):

$$\min_{z\in\mathcal{R}} \frac{1}{\tau}\bar{S}^T H\bar{S} + \frac{1}{2}z^T W(H,G,\mu)z + b^T(H,G)z, \tag{5}$$

where $z \stackrel{\text{def}}{=} (x_1, x_2, \ldots, x_{N-1})$. The $m(N-1)\times m(N-1)$ symmetric tridiagonal block Toeplitz matrix $W(H,G,\mu)$, and the $m(N-1)$-vector $b(H,G)$ are defined as below:

$$W(H,G,\mu) \stackrel{\text{def}}{=} \begin{pmatrix} L+L^T & -\Theta^T & 0 & \ldots & 0 \\ -\Theta & L+L^T & -\Theta^T & \ldots & 0 \\ 0 & -\Theta & L+L^T & & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & L+L^T \end{pmatrix},$$

$$b(H,G) \stackrel{\text{def}}{=} \begin{pmatrix} -\Theta\bar{S} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Here,

$$L \stackrel{\text{def}}{=} \frac{2}{\tau}H - G + \mu\tau C, \quad \text{and} \quad \Theta \stackrel{\text{def}}{=} \frac{1}{\tau}\big(H+H^T\big) - G. \tag{6}$$

Subsequently, we refer to Θ as the *combined impact matrix*. Clearly,

$$L + L^T = \frac{2}{\tau}\big(H+H^T\big) - \big(G+G^T\big) + 2\mu\tau C = \big(\Theta+\Theta^T\big) + 2\mu\tau C.$$

Lemma 2.1 of [42] states that positive (semi)definiteness of Θ is a necessary and sufficient condition for the positive (semi)definiteness of $W(H,G,0)$, when the matrix G is symmetric. Furthermore, this condition is sufficient for the positive (semi)definiteness of $W(H,G,\mu)$, for any $\mu \geq 0$.

In problem (5), $\mathcal{R}$ denotes the set of feasible execution strategies. When purchasing is allowed during a sell execution and no other constraint is imposed, $\mathcal{R} = \mathcal{R}_0 \stackrel{\text{def}}{=} \mathbb{R}^{m(N-1)}$. The set $\mathcal{R}$ may also include constraints on the asset positions. For example, a liquidation plan may prohibit purchasing over the trading horizon. In this case, the feasible set $\mathcal{R} = \mathcal{R}_c$, where

$$\mathcal{R}_c \stackrel{\text{def}}{=} \big\{z = (x_1, x_2, \ldots, x_{N-1}) \in \mathbb{R}^{m(N-1)} : \bar{S} \geq x_1,\ x_{k-1} \geq x_k \text{ for } k = 2, \ldots, N\big\}. \tag{7}$$

Optimal portfolio execution in a continuous-time framework has also been studied, mostly to trade a single asset, see, e.g., [26, 50], and the references therein. Gökay et al. [29] provide a survey on several discrete and continuous time models for the optimal portfolio execution. The impact of different market price dynamics and the choice of static and dynamic strategies have been discussed in the literature, see, e.g., [2, 28, 43].

In modeling the optimal portfolio execution problem, one of the main challenges is to estimate the price impact parameters (see, e.g., [54] or [11]). In addition, it has been shown in [42] that estimation errors in the impact matrices may severely affect the optimal execution strategy and the efficient frontier, especially when $\lambda_{\min}(W(H, G, \mu))$ is small. Here, $\lambda_{\min}(\cdot)$ denotes the smallest eigenvalue of a matrix. This sensitivity may prevent practical applicability of the optimal execution strategy from the nominal optimal portfolio execution problem (5).

This estimation risk in impact matrices has often been ignored in the current literature on the optimization formulation for determining an optimal execution strategy. Given that estimated impact matrices are inevitably inaccurate, this motivates us to devise an optimization method next, in which this uncertainty is explicitly taken into account.

3 Classical robust optimization

Robust optimization has been broadly used in various fields [5, 12], with portfolio management as one of its main applications, see, e.g., [6, 13, 16, 17, 20, 23, 27, 38, 41, 51, 56] and the references therein. In this methodology, data uncertainty is described by an uncertainty set, which hopefully includes all or most possible realizations of the uncertain input parameters. Given a nonempty, convex, and compact uncertainty set $\mathcal{U}$, robust optimization yields a solution that optimizes the worst-case performance when the input data belongs to $\mathcal{U}$.

An uncertainty set is typically specified by a confidence interval associated with a statistical method to estimate the parameters based on historical data, see, e.g., [30]. Its specification may depend on the desired level of robustness and assumptions about modeling errors. The choice of the uncertainty set also contributes to tractability and conservativeness of the approach. Intervals and ellipsoids have typically been used in the literature on robust optimization to describe an uncertainty set.

We explore here the usefulness of the robust optimization for the optimal portfolio execution problem (3) with uncertain impact matrices, henceforth denoted by $\tilde{H}$ and $\tilde{G}$. Subsequently, $(\tilde{H}, \tilde{G})$ denotes a vector in $\mathbb{R}^{2m^2}$, obtained by stacking the columns of the matrices $\tilde{H}$ and $\tilde{G}$ on top of one another. Since covariance matrix can be estimated relatively more accurately, in comparison to the impact matrices, we continue to assume that the covariance matrix C is accurately given.

Let $\mathcal{U} \subseteq \mathbb{R}^{2m^2}$ denote a compact uncertainty set for impact matrices, a robust optimal execution strategy can be obtained by solving the following *robust counterpart problem*:

$$RC(\mathcal{U}): \quad \inf_{z \in \mathcal{R}} \max_{(\tilde{H}, \tilde{G}) \in \mathcal{U}} \frac{1}{\tau} \bar{S}^T \tilde{H} \bar{S} + \frac{1}{2} z^T W(\tilde{H}, \tilde{G}, \mu) z + b^T(\tilde{H}, \tilde{G}) z.$$

Compactness of $\mathcal{U}$ implies that the optimal value of the inner maximization problem is attained and the use of max rather than sup is justified. Notice that here the uncertainty only affects the objective function.

As the size of the uncertainty set $\mathcal{U}$ increases, the objective value at a robust solution is likely to increase. This drawback of robust optimization has been frequently referred in the literature as the conservativeness of the methodology. Ben-Tal and Nemirovski [7–9], El Ghaoui and Lebret [19], and El Ghaoui et al. [21] suggest to rectify the over-conservatism of robust solutions by specifying an interval uncertainty set to be an ellipsoid of a smaller size. Bertsimas and Sim [10] propose the use of a different subset of the uncertainty set to control the level of conservatism in the robust solution.

In addition to being a conservative approach, specification of an uncertainty set is arbitrary to a large degree, and an uncertainty set built on the historical data may not be able to accurately explain future scenarios. Robust optimization can be viewed as a black box which takes the uncertainty set as its input, and produces a robust solution as an output. Thus, it is important to understand how stable the robust solution is with respect to variation in the uncertainty set.

We say a general robust optimization scheme or a robust solution is *stable* with respect to the uncertainty set, if a small variation in the uncertainty set produces a small change in the robust optimal solution. Next we use a robust optimal portfolio execution example to illustrate potential instability of the robust solution with respect to change in the uncertainty set. Due to its simplicity, in our example, we use an interval uncertainty set $\mathcal{U} = \mathcal{U}_H \times \mathcal{U}_G$, where

$$\begin{aligned} \mathcal{U}_H &\stackrel{\text{def}}{=} [\underline{H}, \overline{H}] = \left\{\tilde{H} \in \mathbb{R}^{m^2} : \underline{H}_{ij} \le \tilde{H}_{ij} \le \overline{H}_{ij}\right\}, \\ \mathcal{U}_G &\stackrel{\text{def}}{=} [\underline{G}, \overline{G}] = \left\{\tilde{G} \in \mathbb{R}^{m^2} : \underline{G}_{ij} \le \tilde{G}_{ij} \le \overline{G}_{ij}\right\}. \end{aligned} \tag{8}$$

For the interval uncertainty set $\mathcal{U} = \mathcal{U}_H \times \mathcal{U}_G$ in (8), the inner maximization problem in $RC(\mathcal{U})$ becomes[1]

$$\begin{aligned} \max_{\tilde{H},\tilde{G}} \quad & \sum_{k=1}^{N}\sum_{i,j=1}^{m} \tilde{G}_{ij}(x_k)_i(x_{k-1}-x_k)_j + \frac{1}{\tau}\sum_{k=1}^{N}\sum_{i,j=1}^{m} \tilde{H}_{ij}(x_k-x_{k-1})_i(x_k-x_{k-1})_j \\ & + \mu\tau\sum_{k=1}^{N} x_k^T C x_k \\ \text{s.t.} \quad & \underline{H}_{ij} \le \tilde{H}_{ij} \le \overline{H}_{ij}, \quad i=1,2,\ldots,m, \ j=1,2,\ldots,m, \\ & \underline{G}_{ij} \le \tilde{G}_{ij} \le \overline{G}_{ij}, \quad i=1,2,\ldots,m, \ j=1,2,\ldots,m, \end{aligned} \tag{9}$$

which is a linear optimization problem in terms of the variables $\tilde{H}_{ij}$ and $\tilde{G}_{ij}$ with box constraints. At the solution, each variable equals either its upper bound or lower bound, depending on the sign of its coefficient in the objective function. Whence a robust solution of problem $RC(\mathcal{U})$ solves the following problem:

[1] The second summation of the objective function in problem (9), at $k = 1$, yields the term $\frac{1}{\tau}\bar{S}^T\tilde{H}\bar{S}$.

$$\inf_{z=(x_1,\ldots,x_{N-1})\in\mathcal{R}} \quad \frac{1}{\tau}\bar{S}^T\overline{H}\bar{S}+\frac{1}{2}z^T W(\overline{H},\overline{G},\mu)z+b^T(\overline{H},\overline{G})z$$
$$+\sum_{i,j}(\overline{G}_{ij}-\underline{G}_{ij})\max\left\{0,\sum_{k=1}^{N-1}(x_k)_i(x_k-x_{k-1})_j\right\}$$
$$+\frac{1}{\tau}\sum_{i,j}(\overline{H}_{ij}-\underline{H}_{ij})$$
$$\times\max\left\{0,-\sum_{k=1}^{N}(x_k-x_{k-1})_i(x_k-x_{k-1})_j\right\}. \qquad (10)$$

This problem can be formulated as minimizing a quadratic function subject to quadratic constraints; an optimization method is not guaranteed to yield a global solution in general.

To understand sensitivity of a robust optimal execution strategy to variation in the interval uncertainty set, we conduct a sensitivity analysis based on simulations; this technique has been previously used for the Markowitz mean variance portfolio optimization, see, e.g., [15]. We assume that there exists an uncertainty set $\mathcal{U}$ which yields a robust strategy with the desired properties; we refer to this as the original uncertainty set. Suppose this uncertainty set is unknown; some perturbed uncertainty set $\overline{\mathcal{U}}$ is instead applied by the decision maker.

The performance of a strategy is represented by a mean-variance efficient frontier. An *original efficient frontier* depicts the performance of the strategy with respect to the original data. An *actual efficient frontier* describes the actual performance (using the original data) of a strategy determined using perturbed data. For given nominal impact matrices H and G, we refer to the solution of problem (5) as the nominal optimal execution strategy. The *original nominal frontier* is the curve of the original mean and variance of the execution cost associated with nominal optimal strategy from the original data when the risk aversion parameter μ varies in $(0,\infty)$. For the perturbed impact matrices $H+\Delta H$ and $G+\Delta G$, the *actual nominal frontier* is the curve of the mean and variance of the execution cost computed from the original nominal impact matrices H and G for the optimal execution strategy determined from the perturbed impact matrices $H+\Delta H$ and $G+\Delta G$.

Similarly we can consider a robust efficient frontier of the robust solution with respect to an uncertainty set $\mathcal{U}$; it is the curve of the worst case mean and variance of the execution cost of the robust solution. The notion of robust efficient frontier is described in [38]. We also extend the notions of *original* and *actual* (mean-variance) efficient frontier to the robust frontier. The *original robust frontier* corresponds to the worst case mean and variance of the execution cost with respect to the original uncertainty set $\mathcal{U}$ for the robust solution obtained from $\mathcal{U}$. An *actual robust frontier* for the perturbed uncertainty set $\overline{\mathcal{U}}$ is the curve of the worst case mean and variance with respect to the original uncertainty set $\mathcal{U}$ for the robust solution computed from a perturbed uncertainty set $\overline{\mathcal{U}}$.

Using simulations, we consider a three asset robust optimal portfolio execution problem with respect to an interval uncertainty set to illustrate sensitivity of the robust solution to the uncertainty set specification; the details are described in Example 3.1.

In our simulation study, we use the open-source solver Gloptipoly3 [36] to compute a solution for problem (10). Gloptipoly3 returns a flag, indicating whether the obtained solution is global or not. Perturbations are selected when Gloptipoly3 has indeed obtained a global solution for the robust optimization problem. This example has been used in [42] to illustrate sensitivity of the nominal execution strategy to the impact matrices; we also include nominal efficient frontiers and nominal solutions here to compare them with the robust efficient frontiers and robust solutions.

Example 3.1 Consider liquidation of three assets with the initial holding $\bar{S}_i = 10^5$, $i = 1, 2, 3$, shares in five days by trading daily, i.e., $T = 5$, $N = 5$, and $\tau = 1$. We assume that there is no constraint on the execution strategy, i.e., $\mathcal{R} = \mathcal{R}_0$. The assets are currently traded at price $P_0 = 50\$/\text{share}$. Let the daily asset price covariance matrix be:

$$C = \begin{pmatrix} 0.3246 & 0.0230 & 0.4204 \\ 0.0230 & 0.0499 & 0.0192 \\ 0.4204 & 0.0192 & 0.7641 \end{pmatrix} \times 1\ \%.$$

The nominal permanent and temporary impact matrices are assumed to be as below[2]:

$$H = 10^{-5} \cdot C \quad \text{and} \quad G = 0.5 \times 10^{-5} \cdot C \tag{11}$$

Note that $\lambda_{\min}(W(H, G, 0)) = 2.5960 \times 10^{-9}$.

For simplicity we assume that the temporary impact matrix is accurately given, i.e., $\overline{H} = \underline{H} = H$, and only the permanent impact matrix is uncertain with $\overline{G} = 3 \cdot G$ and $\underline{G} = 0.2 \cdot G$, i.e.,

$$\mathcal{U}_H = \{H\}, \qquad \mathcal{U}_G = [0.2 \cdot G,\ 3 \cdot G] \tag{12}$$

Notice that $\lambda_{\min}(W(\overline{H}, \overline{G}, 0)) = 8.6534 \times 10^{-10}$, which is smaller than $\lambda_{\min}(W(H, G, 0))$.

We now add 5 % perturbation $\Delta G^{(\ell)}$ and $\Delta \overline{G}^{(\ell)}$ to the nominal permanent impact matrix G and the upper bound of the original uncertainty set $\mathcal{U}_G$ as follows:

$$\Delta G^{(\ell)} = 5\ \% \cdot \max_{i,j}\{|G_{ij}|\}\phi^{(\ell)}, \qquad \Delta \overline{G}^{(\ell)} = 5\ \% \cdot \max_{i,j}\{|\overline{G}_{ij}|\}\phi^{(\ell)}, \tag{13}$$

where $\phi^{(\ell)}$ is a standard normal random sample (computed using randn in MATLAB). A sample $\phi^{(\ell)}$ is selected only if the nominal solution corresponding to the perturbed permanent impact matrix $G + \Delta G^{(\ell)}$ uniquely exits (the matrix $W(H, G + \Delta G^{(\ell)}, 0)$ is positive definite), the perturbed uncertainty set $\overline{\mathcal{U}}_G = [\underline{G},\ \overline{G} + \Delta \overline{G}^{(\ell)}]$ is a valid interval (all entries of the matrix $\overline{G} + \Delta \overline{G}^{(\ell)} - \underline{G}$ are nonnegative), and Gloptipoly3 obtains a global solution for the robust optimization problem (10) with $\overline{\mathcal{U}} = \mathcal{U}_H \times \overline{\mathcal{U}}_G$.

The original robust frontier and actual robust frontiers corresponding to 50 perturbations $\overline{\mathcal{U}}_G$ to the original uncertainty set $\mathcal{U} = \mathcal{U}_H \times \mathcal{U}_G$ (with $\mu \in [0, 10^{-5}]$) are

[2] The units for H and G are \$ per share^2 and \$ per day per share^2, respectively.

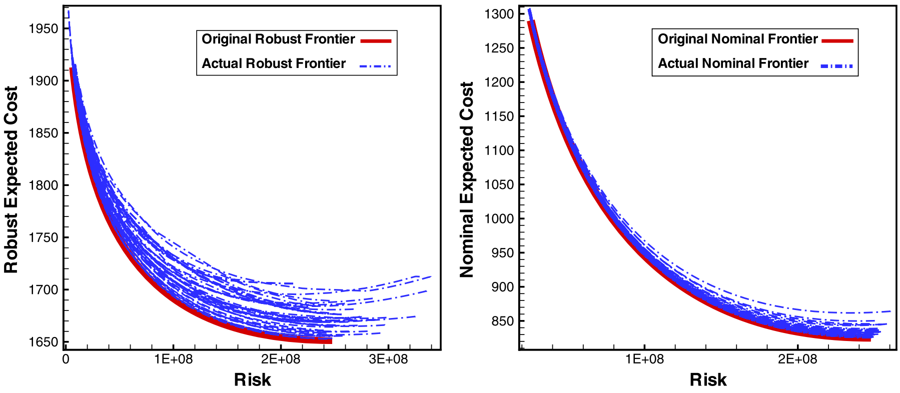

Fig. 1 Comparing sensitivity of the robust efficient frontier to 5 % perturbation in the upper bound of the uncertainty set $\mathcal{U}_G$ with sensitivity of the nominal efficient frontier to 5 % perturbation in the nominal permanent impact matrix G

graphed in the left plot in Fig. 1. We observe large deviations of the actual robust frontiers from the original robust frontier. This indicates that the robust frontier can be unstable to perturbations in the uncertainty set. For comparison, the right plot in Fig. 1 graphs the original nominal frontier and 50 actual nominal frontiers corresponding to 50 perturbed nominal impact matrices $G + \Delta G^{(\ell)}$. This plot shows that sensitivity of the robust frontiers to perturbations in the uncertainty set may be larger than sensitivity of the nominal frontiers to perturbations in the nominal impact matrices.

In addition, it can be observed from Fig. 1 that deviations of actual frontiers from the original ones are more prominent for small risk aversion parameters. We further examine variation in the optimal execution strategy when $\mu = 0$. Figure 2 illustrates sensitivity of the robust optimal execution strategy for $\mu = 0$ to perturbations in $\mathcal{U}_G$ (left plots) and compares it to sensitivity of the nominal solution to perturbations in the nominal permanent impact matrix G (right plots). Significant variation in the robust optimal execution strategy can be observed from the left plots; variation is more severe in comparison to variation in the nominal optimal execution strategy depicted in the right plots. Note that both the original nominal solution and the original robust solution in this case are the naive strategy ($n_k = \frac{1}{N}\bar{S}$, for $k = 1, \ldots, N$), since the matrices G and $\overline{G}$ are symmetric, see Proposition 2.1 in [42].

Example 3.1 clearly illustrates that the robust optimal execution strategy can be unstable with respect to variation in the uncertainty set. This can also be seen when the set of feasible execution strategies is $\mathcal{R}_c$. In this case, for every $z = (x_1, \ldots, x_{N-1}) \in \mathcal{R}_c$, $x_{k-1} \geq x_k \geq 0$. Hence, $(\overline{H}, \overline{G})$ is the solution to problem (9) and the worst case realization of impact matrices is the same regardless which execution strategy is adopted. Therefore, when $W(\overline{H}, \overline{G}, 0)$ is positive definite, the (global) robust solution of problem $RC(\mathcal{U})$ with respect to the interval uncertainty set $\mathcal{U} = \mathcal{U}_H \times \mathcal{U}_G$ can be obtained simply by solving the following convex quadratic

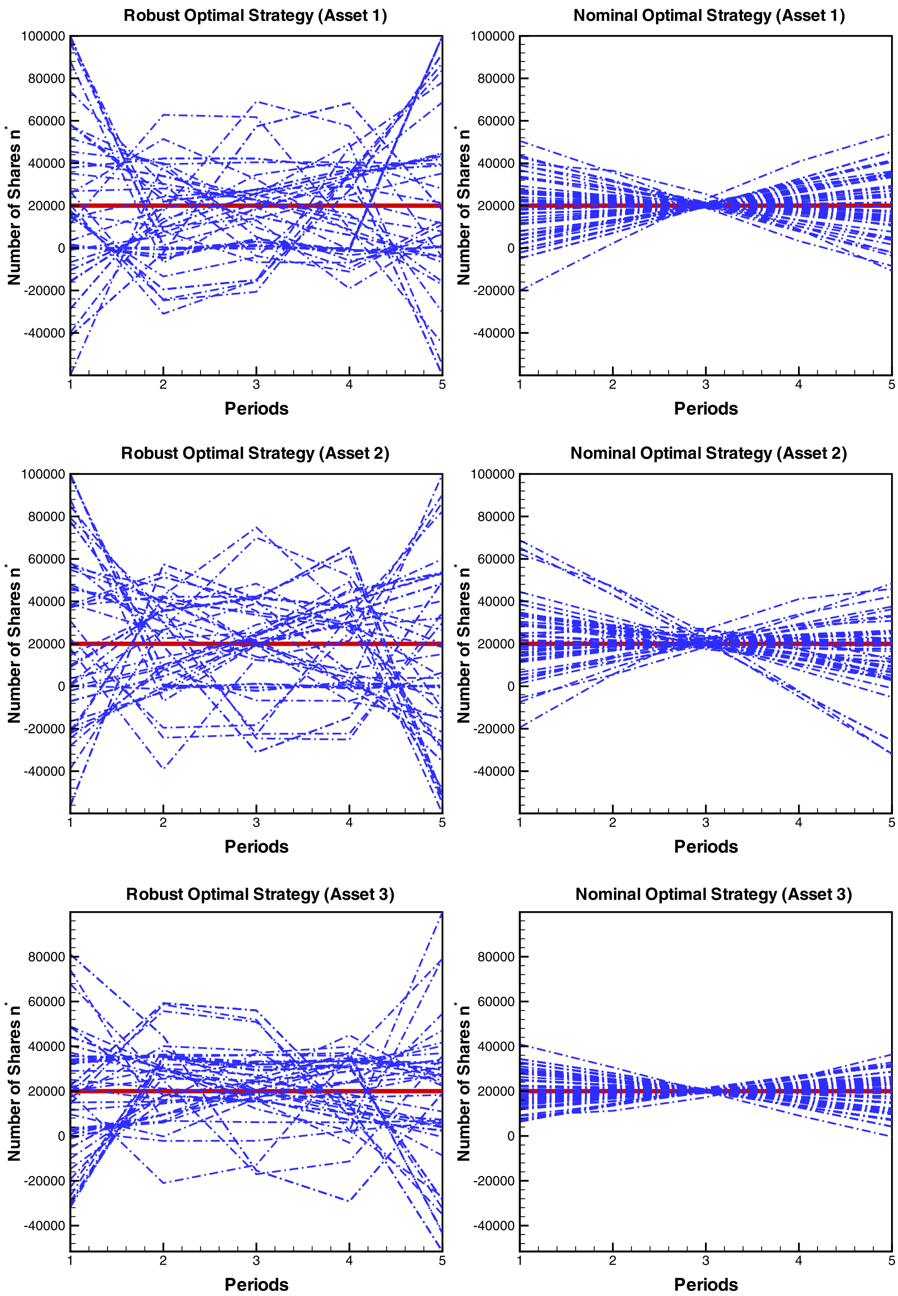

Fig. 2 Comparing sensitivity of the (classical) robust solution to 5 % perturbation in the upper bound of the uncertainty set $\mathcal{U}_G$ with sensitivity of the nominal solution to 5 % perturbation in the nominal permanent impact matrix G. Risk aversion parameter is $\mu = 0$ and $\mathcal{R} = \mathcal{R}_0$

programming problem:

$$\min_{z\in\mathcal{R}_c} \quad \frac{1}{\tau}\bar{S}^T\bar{H}\bar{S} + \frac{1}{2}z^T W(\overline{H},\overline{G},\mu)z + z^T b(\overline{H},\overline{G}). \tag{14}$$

Consequently, applying the robust optimization approach to obtain a robust execution strategy, we end up with a nominal optimal portfolio execution problem with the impact matrices replaced by the upper bounds of the uncertainty set. Hence, sensitivity of the robust solution to variation in the uncertainty set $\mathcal{U} = \mathcal{U}_H \times \mathcal{U}_G$ is the same as sensitivity of the solution to perturbation in the impact matrices $\overline{H}$ and $\overline{G}$. Applying Theorem 3.1 in [42] for the optimal portfolio execution problem (14) implies that the robust solution may be very sensitive to change in the uncertainty set $\mathcal{U}$ if $\lambda_{\min}(W(\overline{H},\overline{G},\mu))$ is sufficiently small. Indeed, this sensitivity may be larger than the sensitivity of the nominal optimal execution strategy to the nominal impact matrices (H,G) when $\lambda_{\min}(W(\overline{H},\overline{G},\mu)) \le \lambda_{\min}(W(H,G,\mu))$.

Next we show that the robust optimal execution strategy can be computed by semidefinite programming when the Hessian $W(\tilde{H},\tilde{G},\mu)$ is positive definite for every $(\tilde{H},\tilde{G}) \in \mathcal{U}$. Indeed, this method will also be used for our proposed regularized robust optimization described in Sect. 5.

4 Robust optimal execution strategy under convexity

For simplicity, we denote the objective function of $RC(\mathcal{U})$ by $\Upsilon(z,\tilde{H},\tilde{G})$, i.e.,

$$\Upsilon(z,\tilde{H},\tilde{G}) \overset{\text{def}}{=} \frac{1}{\tau}\bar{S}^T\tilde{H}\bar{S} + \frac{1}{2}z^T W(\tilde{H},\tilde{G},\mu)z + b^T(\tilde{H},\tilde{G})z. \tag{15}$$

The function $\Upsilon(z,\tilde{H},\tilde{G})$ is linear in $(\tilde{H},\tilde{G})$ and quadratic in z. The function $\Upsilon(\cdot,\tilde{H},\tilde{G})$ is in general non-convex, as the uncertainty set $\mathcal{U}$ may include scenarios $(\tilde{H},\tilde{G})$ where the matrix $W(\tilde{H},\tilde{G},\mu)$ is not positive semidefinite. Thus robust problem $RC(\mathcal{U})$ is NP-hard in general.[3]

When $W(\tilde{H},\tilde{G},\mu)$ is positive semidefinite, for every $(\tilde{H},\tilde{G}) \in \mathcal{U}$, $\Upsilon(\cdot,\tilde{H},\tilde{G})$ is a convex quadratic function. Using Theorem 5.5 of [46], the function $\max_{(\tilde{H},\tilde{G})\in\mathcal{U}} \Upsilon(z,\tilde{H},\tilde{G})$ is convex in z. Thus, the problem $(\inf_{z\in\mathcal{R}}(\max_{(\tilde{H},\tilde{G})\in\mathcal{U}} \Upsilon(z,\tilde{H},\tilde{G})))$ is a convex optimization problem for a convex feasible set $\mathcal{R}$.

The following proposition shows the existence and uniqueness of the saddle point for the minimax problem $RC(\mathcal{U})$ under convexity assumption.

Proposition 4.1 *Let $\mathcal{R}$ be nonempty, convex, and closed, and the uncertainty set $\mathcal{U}$ be nonempty, convex, and compact. For a given risk aversion parameter $\mu \ge 0$, assume that the matrix $W(\tilde{H},\tilde{G},\mu)$ is positive definite, for every $(\tilde{H},\tilde{G}) \in \mathcal{U}$. Then the minimax problem $RC(\mathcal{U})$ has a saddle point (z_u, H_u, G_u), i.e.,*

$$\Upsilon(z_u,\tilde{H},\tilde{G}) \le \Upsilon(z_u,H_u,G_u) \le \Upsilon(z,H_u,G_u), \quad \forall(\tilde{H},\tilde{G}) \in \mathcal{U},\ \forall z \in \mathcal{R}. \tag{16}$$

[3]Note that the problem of minimizing a non-convex quadratic function is known to be NP-hard [45].

Moreover, for every two saddle points $(z^{(1)}, H^{(1)}, G^{(1)})$ *and* $(z^{(2)}, H^{(2)}, G^{(2)})$*, we have* $z^{(1)} = z^{(2)}$*, i.e., the robust optimal execution strategy from problem RC*($\mathcal{U}$) *is unique.*

Proof From the convexity of $\mathcal{R}$ and $\mathcal{U}$, compactness of $\mathcal{U}$, and $\Upsilon(z, \tilde{H}, \tilde{G})$ being strictly convex in z and linear in $(\tilde{H}, \tilde{G})$, we have

$$\inf_{z\in\mathcal{R}} \max_{(\tilde{H},\tilde{G})\in\mathcal{U}} \Upsilon(z, \tilde{H}, \tilde{G}) = \max_{(\tilde{H},\tilde{G})\in\mathcal{U}} \inf_{z\in\mathcal{R}} \Upsilon(z, \tilde{H}, \tilde{G}), \tag{17}$$

see, e.g., Theorem 3 of [48].

Let $(H_u, G_u) \in \mathcal{U}$ be an optimal point for the outer maximization problem below

$$\max_{(\tilde{H},\tilde{G})\in\mathcal{U}} \inf_{z\in\mathcal{R}} \quad \Upsilon(z, \tilde{H}, \tilde{G}).$$

Thus,

$$\left(\inf_{z\in\mathcal{R}} \Upsilon(z, H_u, G_u)\right) = \left(\max_{(\tilde{H},\tilde{G})\in\mathcal{U}} \inf_{z\in\mathcal{R}} \Upsilon(z, \tilde{H}, \tilde{G})\right). \tag{18}$$

Since $(H_u, G_u) \in \mathcal{U}$, $W(H_u, G_u, \mu)$ is positive definite. Thus $\Upsilon(z, H_u, G_u)$ is a strictly convex quadratic function. Since $\mathcal{R}$ is closed, there exists $z_{(H_u,G_u)} \in \mathcal{R}$ at which $\inf_{z\in\mathcal{R}} \Upsilon(z, H_u, G_u)$ is uniquely attained (see, e.g., Proposition 2.5 of [18]). Thus

$$\Upsilon(z_{(H_u,G_u)}, H_u, G_u) = \inf_{z\in\mathcal{R}} \Upsilon(z, H_u, G_u) = \max_{(\tilde{H},\tilde{G})\in\mathcal{U}} \inf_{z\in\mathcal{R}} \Upsilon(z, \tilde{H}, \tilde{G}), \tag{19}$$

and

$$\Upsilon(z_{(H_u,G_u)}, H_u, G_u) \le \Upsilon(z, H_u, G_u), \quad \forall z \in \mathcal{R}. \tag{20}$$

From (19) and (17), we get

$$\inf_{z\in\mathcal{R}} \max_{(\tilde{H},\tilde{G})\in\mathcal{U}} \Upsilon(z, \tilde{H}, \tilde{G}) = \Upsilon(z_{(H_u,G_u)}, H_u, G_u).$$

Therefore, $z_{(H_u,G_u)}$ is a solution of the outer infimum on the left problem of equation (17). Thus

$$\Upsilon(z_{(H_u,G_u)}, H_u, G_u) = \inf_{z\in\mathcal{R}} \max_{(\tilde{H},\tilde{G})\in\mathcal{U}} \Upsilon(z, \tilde{H}, \tilde{G}) = \max_{(\tilde{H},\tilde{G})\in\mathcal{U}} \Upsilon(z_{(H_u,G_u)}, \tilde{H}, \tilde{G}).$$

Hence,

$$\Upsilon(z_{(H_u,G_u)}, \tilde{H}, \tilde{G}) \le \Upsilon(z_{(H_u,G_u)}, H_u, G_u), \quad \forall (\tilde{H}, \tilde{G}) \in \mathcal{U}. \tag{21}$$

Inequalities (20) and (21) imply that $(z_{(H_u,G_u)}, H_u, G_u)$ is a saddle point.

For the uniqueness, note that positive definiteness of $W(\tilde{H}, \tilde{G}, \mu)$ for every $(\tilde{H}, \tilde{G}) \in \mathcal{U}$ yields strict convexity of $\Upsilon(\cdot, \tilde{H}, \tilde{G})$. In particular, $\Upsilon(\cdot, H^{(1)}, G^{(1)})$ is strictly convex, and the problem $\min_{z \in \mathcal{R}} \Upsilon(z, H^{(1)}, G^{(1)})$ has a unique solution. Hence, if $z^{(1)} \neq z^{(2)}$,

$$\Upsilon\big(z^{(1)}, H^{(1)}, G^{(1)}\big) < \Upsilon\big(z^{(2)}, H^{(1)}, G^{(1)}\big) \le \Upsilon\big(z^{(2)}, H^{(2)}, G^{(2)}\big).$$

This contradicts to the fact that both $(z^{(1)}, H^{(1)}, G^{(1)})$ and $(z^{(2)}, H^{(2)}, G^{(2)})$ are saddle points and consequently $\Upsilon(z^{(1)}, H^{(1)}, G^{(1)}) = \Upsilon(z^{(2)}, H^{(2)}, G^{(2)})$. Therefore, there must be $z^{(1)} = z^{(2)}$. □

Proposition 4.1 indicates that the robust optimal execution strategy is unique, when the matrix $W(\tilde{H}, \tilde{G}, \mu)$ is positive definite for every $(\tilde{H}, \tilde{G}) \in \mathcal{U}$.

A typical approach to obtain a robust solution is to find a semidefinite programming (SDP) representation for the robust counterpart problem $RC(\mathcal{U})$. Ben-Tal and Nemirovski [7] show that the robust counterpart of an uncertain convex quadratically constrained quadratic programming problem, with separate ellipsoidal uncertainty sets for the Hessian and linear term of the objective function can be explicitly modeled as a linear semidefinite programming. As is explained in [34], the model in [7] places the uncertainty description on the square root of the Hessian, whence, every matrix in the uncertainty set is positive semidefinite. However, when one has an uncertainty description for only the Hessian, transferring that into an uncertainty description on the Cholesky-like factors can be difficult. Ben-Tal and Nemirovski [7] further discuss that a more general uncertainty for the Hessian and linear term of the quadratic objective function leads to an NP-hard robust counterpart problem.

Here, we apply semidefinite programming to solve problem $RC(\mathcal{U})$, when the matrix $W(\tilde{H}, \tilde{G}, \mu)$ is positive definite for every $(\tilde{H}, \tilde{G}) \in \mathcal{U}$. However, in contrast to the typical approach, see, e.g., [5], in which the dual of the inner maximization problem is taken, similar to Kim and Boyd [39], we first switch the order of min and max; then we take the dual of the minimization problem and show that it is SDP representable. We summarize our discussion in the following proposition. Below, we assume that the set of feasible execution strategies is defined by linear inequality constraints

$$\mathcal{R} = \big\{z \in \mathbb{R}^{m(N-1)} : Az \le c\big\}, \tag{22}$$

where c is an r-vector and A is a $r \times m(N-1)$ matrix. Considering $\mathcal{R}$ as in (22) allows us to treat any linear inequality constraint such as nonnegativity constraints or bound constraints on execution strategies, in a unified manner. Furthermore, since an equality constraint can be represented using two inequality constraints, it can also be used when linear equality constraints are imposed on an execution strategy.

Proposition 4.2 *Let the uncertainty set $\mathcal{U}$ be nonempty, convex, and compact, and the matrix $W(\tilde{H}, \tilde{G}, \mu)$ be positive definite for every $(\tilde{H}, \tilde{G}) \in \mathcal{U}$. Furthermore, assume the nonempty feasible set $\mathcal{R}$ is as in (22). Then the robust solution to $RC(\mathcal{U})$ equals*

$$z_u = -W(H_u, G_u, \mu)^{-1}\big(b(H_u, G_u) + A^T \lambda_u\big), \tag{23}$$

where (H_u, G_u) and $\lambda_u \in \mathbb{R}^r_+$ constitute a solution of the following problem:

$$P(\mathcal{U}): \quad \max_{(\tilde{H},\tilde{G})\in\mathcal{U},\lambda\in\mathbb{R}_+,v\in\mathbb{R}} \quad v \tag{24}$$
$$\text{s.t.} \quad \begin{bmatrix} \frac{2}{\tau}\bar{S}^T\tilde{H}\bar{S} - 2c^T\lambda - 2v & (b(\tilde{H},\tilde{G}) + A^T\lambda)^T \\ b(\tilde{H},\tilde{G}) + A^T\lambda & W(\tilde{H},\tilde{G},\mu) \end{bmatrix} \succeq 0.$$

When no constraint is imposed, i.e., $\mathcal{R} = \mathcal{R}_0$*, the robust solution of* $RC(\mathcal{U})$ *is*

$$z_u = -W(H_u, G_u, \mu)^{-1} b(H_u, G_u), \tag{25}$$

where (H_u, G_u) *constitutes an optimal point of the following problem*:

$$\max_{(\tilde{H},\tilde{G})\in\mathcal{U},v\in\mathbb{R}} \quad v \tag{26}$$
$$s.t. \quad \begin{bmatrix} \frac{2}{\tau}\bar{S}^T\tilde{H}\bar{S} - 2v & b(\tilde{H},\tilde{G})^T \\ b(\tilde{H},\tilde{G}) & W(\tilde{H},\tilde{G},\mu) \end{bmatrix} \succeq 0.$$

Proof The given assumptions and Proposition 4.1 imply that the infimum is attained. Furthermore, problem $RC(\mathcal{U})$ equals:

$$\max_{(\tilde{H},\tilde{G})\in\mathcal{U}} \frac{1}{\tau}\bar{S}^T\tilde{H}\bar{S} + \min_{z\in\mathcal{R}}\left(\frac{1}{2}z^T W(\tilde{H},\tilde{G},\mu)z + b^T(\tilde{H},\tilde{G})z\right). \tag{27}$$

The Lagrangian function of the inner minimization problem in (27) is:

$$\begin{aligned} L(z,\lambda) &= \frac{1}{2}z^T W(\tilde{H},\tilde{G},\mu)z + b^T(\tilde{H},\tilde{G})z + \lambda^T(Az - c) \\ &= \frac{1}{2}z^T W(\tilde{H},\tilde{G},\mu)z + \big(b(\tilde{H},\tilde{G}) + A^T\lambda\big)^T z - c^T\lambda. \end{aligned}$$

Since $L(z,\lambda)$ is a strictly convex quadratic function of z, the Lagrange dual problem is:

$$\begin{aligned} &\max_{\lambda\in\mathbb{R}^r_+}\Big(\min_{z\in\mathbb{R}^{m(N-1)}} L(z,\lambda)\Big) \\ &\quad = \max_{\lambda\in\mathbb{R}^r_+}\left(-\frac{1}{2}\big(b(\tilde{H},\tilde{G}) + A^T\lambda\big)^T W(\tilde{H},\tilde{G},\mu)^{-1}\big(b(\tilde{H},\tilde{G}) + A^T\lambda\big) - c^T\lambda\right). \end{aligned} \tag{28}$$

Here $\mathbb{R}^r_+$ denotes the nonnegative orthant. Since $\mathcal{R}$ is defined by linear inequalities, Slater's condition and consequently strong duality hold for the inner minimization problem of (27). Thus:

$$\min_{z\in\mathcal{R}}\left(\frac{1}{2}z^T W(\tilde{H},\tilde{G},\mu)z + b^T(\tilde{H},\tilde{G})z\right) = \max_{\lambda\in\mathbb{R}^r_+}\Big(\min_{z\in\mathbb{R}^{m(N-1)}} L(z,\lambda)\Big).$$

Thus problem (27) is reduced to:

$$\begin{aligned} \max_{(\tilde{H},\tilde{G})\in\mathcal{U},\lambda\in\mathbb{R}^r_+} \quad & \frac{1}{\tau}\bar{S}^T\tilde{H}\bar{S} - \frac{1}{2}\big(b(\tilde{H},\tilde{G}) + A^T\lambda\big)^T \\ & \times W^{-1}(\tilde{H},\tilde{G},\mu)\big(b(\tilde{H},\tilde{G}) + A^T\lambda\big) - c^T\lambda. \end{aligned} \tag{29}$$

Problem (29) can be reformulated as:

$$\begin{aligned} \max_{(\tilde{H},\tilde{G})\in\mathcal{U},\lambda\in\mathbb{R}^r_+,v\in\mathbb{R}} \quad & v \\ \text{s.t.} \quad & \frac{1}{\tau}\bar{S}^T\tilde{H}\bar{S} - \frac{1}{2}\big(b(\tilde{H},\tilde{G}) + A^T\lambda\big)^T W(\tilde{H},\tilde{G},\mu)^{-1} \\ & \times \big(b(\tilde{H},\tilde{G}) + A^T\lambda\big) - c^T\lambda \geq v. \end{aligned}$$

Since $W(\tilde{H},\tilde{G},\mu)$ is positive definite, using the Schur complement, inequality

$$\frac{1}{\tau}\bar{S}^T\tilde{H}\bar{S} - \frac{1}{2}\big(b(\tilde{H},\tilde{G}) + A^T\lambda\big)^T W(\tilde{H},\tilde{G},\mu)^{-1}\big(b(\tilde{H},\tilde{G}) + A^T\lambda\big) - c^T\lambda \geq v, \quad (30)$$

holds if and only if the linear matrix inequality

$$\begin{bmatrix} \frac{2}{\tau}\bar{S}^T\tilde{H}\bar{S} - 2c^T\lambda - 2v & (b(\tilde{H},\tilde{G}) + A^T\lambda)^T \\ b(\tilde{H},\tilde{G}) + A^T\lambda & W(\tilde{H},\tilde{G},\mu) \end{bmatrix} \succeq 0,$$

holds, with strict positive definiteness in the last constraint if and only if strict inequality holds in inequality (30).

Therefore, a solution of the inner maximization problem in $RC(\mathcal{U})$ can be obtained by solving the maximization problem $P(\mathcal{U})$. Let the pair (H_u, G_u) and $\lambda_u \in \mathbb{R}^r_+$ be a solution of problem $P(\mathcal{U})$, then the robust optimal strategy equals (23).

When no constraint is imposed, i.e, $\mathcal{R} = \mathcal{R}_0$, problem (27) is reduced to the following problem:

$$\max_{(\tilde{H},\tilde{G})\in\mathcal{U}} \frac{1}{\tau}\bar{S}^T\tilde{H}\bar{S} - \frac{1}{2}b^T(\tilde{H},\tilde{G})W^{-1}(\tilde{H},\tilde{G},\mu)b(\tilde{H},\tilde{G}). \quad (31)$$

A similar discussion then implies that the robust solution becomes (25) where (H_u, G_u) is an optimal point of problem (26). □

At an optimal point of $P(\mathcal{U})$, optimal objective value v represents the execution cost corresponding to the robust optimal execution strategy. Convexity of the objective function, $\mathcal{U}$, and the linear matrix inequality constraint imply that problem $P(\mathcal{U})$ is a convex programming problem. When $\mathcal{U}$ is defined by linear matrix inequalities, problem $P(\mathcal{U})$ is a linear semidefinite programming problem; this problem can be solved using high-quality open-source solvers, e.g., SEDUMI [49] or SDPT3 [53].

An advantage of the above derivation for the robust solution is that this approach does not depend on any specific structure (e.g., interval or ellipsoidal) of the uncertainty set. We also adopt this derivation for the regularized uncertainty set introduced in Sect. 5. It is worth mentioning that formulation $P(\mathcal{U})$ allows us to include the constraint $\tilde{G} = \tilde{G}^T$ in the uncertainty set specification, when there is some evidence that the permanent impact matrix is symmetric.

5 Regularized robust optimization

Example 3.1 illustrates that a robust execution strategy can be sensitive to the uncertainty set specification. Now we propose a regularized robust optimization formulation to address this issue.

For the nominal optimal portfolio execution problem (5), sensitivity of the optimal execution strategy and the efficient frontier has been studied in [42]. This analysis shows that, when the minimum eigenvalue of the Hessian of the objective function $W(H, G, \mu)$ is small, the optimal solution may vary significantly when the impact matrices change slightly. This result suggests that excluding those elements, which yield a small minimum eigenvalue of $W(\tilde{H}, \tilde{G}, 0)$, from the uncertainty set $\mathcal{U}$, may prevent an unstable solution. This idea is also related to the well known regularization technique in which prior information is included in the problem formulation to stabilize the solution. The most common form of regularization for ill-posed least square problems is Tikhonov regularization, see, e.g., [22, 25, 52], where a two-norm bound constraint is included. Here we propose to use a regularized uncertainty set to obtain more stable robust solutions.

Let $\mathcal{U} \subseteq \mathbb{R}^{2m^2}$ be a nonempty, convex, and compact uncertainty set for the impact matrices. Given $\mathcal{U}$ and a positive constant $\rho > 0$, we impose the regularization constraint $\lambda_{\min}(W(\tilde{H}, \tilde{G}, 0)) \geq \rho$ on the uncertainty set:

$$\mathcal{V}(\mathcal{U}, \rho) \stackrel{\text{def}}{=} \big\{(\tilde{H}, \tilde{G}) \in \mathcal{U} \,|\, \lambda_{\min}\big(W(\tilde{H}, \tilde{G}, 0)\big) \geq \rho\big\}. \tag{32}$$

We refer to the parameter ρ and the set $\mathcal{V}(\mathcal{U}, \rho)$ as the *regularization parameter* and the *regularized uncertainty set*, respectively. The regularization constraint $\lambda_{\min}(W(\tilde{H}, \tilde{G}, 0)) \geq \rho$ is equivalent to the matrix inequality constraint $W(\tilde{H}, \tilde{G}, 0) \succeq \rho I_{m(N-1)}$ where $I_{m(N-1)}$ is the $m(N-1) \times m(N-1)$ identity matrix. Figure 3 illustrates how the regularized uncertainty set $\mathcal{V}(\mathcal{U}, \rho)$ compares with $\mathcal{U}$ for two values of ρ in a single asset execution.

We note that convexity of $\mathcal{U}$ and convexity of the regularization constraint imply convexity of $\mathcal{V}(\mathcal{U}, \rho)$. Moreover, since the function $\lambda_{\min}(\cdot)$ is a continuous function,

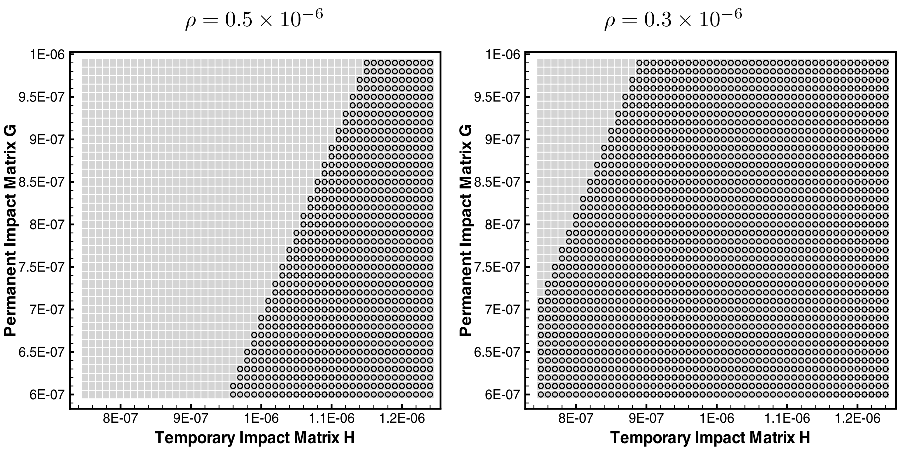

Fig. 3 Regularized uncertainty set $\mathcal{V}(\mathcal{U}, \rho)$ versus uncertainty set $\mathcal{U}$. Here, $m = 1$, $N = 5$, and the nominal impact matrices are $H = 10^{-6}$ and $G = 8 \times 10^{-7}$. The uncertainty set is $\mathcal{U} = \{(\tilde{H}, \tilde{G}) \in \mathcal{U}_H \times \mathcal{U}_G : \lambda_{\min}(W(\tilde{H}, \tilde{G}, 0)) \geq 0\}$ where $\mathcal{U}_H = [0.75 \cdot H,\ 1.25 \cdot H]$ and $\mathcal{U}_G = [0.75 \cdot G,\ 1.25 \cdot G]$. The *grey area* denotes the original uncertainty set and *circle pattern* denotes the regularized uncertainty set $\mathcal{V}(\mathcal{U}, \rho)$

closeness of $\mathcal{U}$ implies closeness of $\mathcal{V}(\mathcal{U},\rho)$. For every $(\tilde{H},\tilde{G}) \in \mathcal{V}(\mathcal{U},\rho)$ and $\mu \geq 0$, the Courant-Fischer Theorem (see, e.g., Theorem 8.1.5 in [31]) yields

$$\lambda_{\min}\big(W(\tilde{H},\tilde{G},\mu)\big) = \lambda_{\min}\big(W(\tilde{H},\tilde{G},0) + 2\mu\tau I_N \otimes C\big) \geq \rho + 2\mu\tau\lambda_{\min}(C), \quad (33)$$

where $\otimes$ denotes the Kronecker product of two matrices. Thus by imposing the regularization constraint, we ensure that the minimum eigenvalue of the Hessian of the objective function at the worst case impact matrices is positive and not very small.

The regularization parameter value ρ affects the size of the regularized uncertainty set $\mathcal{V}(\mathcal{U},\rho)$. If ρ increases, the size of the uncertainty set decreases, i.e., implicitly one is demanding robustness with respect to a smaller set of parameter values. As a result, the resulting robust strategy will be less conservative.

To ensure that the set $\mathcal{V}(\mathcal{U},\rho)$ is nonempty, the regularization parameter ρ needs to be chosen carefully. In particular, when $\lambda_{\min}(W(H,G,0)) > 0$ for the nominal impact matrices $(H,G) \in \mathcal{U}$, the regularized uncertainty set $\mathcal{V}(\mathcal{U},\rho)$ is nonempty for any $\rho \leq \lambda_{\min}(W(H,G,0))$. Thus the regularization parameter can be proportional to this value. If the regularization parameter ρ is strictly greater than $\max_{(\tilde{H},\tilde{G})\in\mathcal{U}} \lambda_{\min}(W(\tilde{H},\tilde{G},0))$, the regularized uncertainty set becomes empty.

Given an uncertainty set $\mathcal{U}$ and a positive regularization parameter ρ, the regularized robust optimization formulation is given below:

$$\Phi_\mu(\rho) \stackrel{\text{def}}{=} \min_{z\in\mathcal{R}} \max_{(\tilde{H},\tilde{G})\in\mathcal{V}(\mathcal{U},\rho)} \frac{1}{\tau}\bar{S}^T\tilde{H}\bar{S} + \frac{1}{2}z^T W(\tilde{H},\tilde{G},\mu)z + b^T(\tilde{H},\tilde{G})z. \quad (34)$$

When z^* constitutes a solution of problem (34), it is subsequently called a *regularized robust optimal execution strategy*. Including a regularization constraint also allows us to compute a regularized robust solution using the semidefinite programming representation and (23) described in Sect. 4. This is the case even for an interval uncertainty set $\mathcal{U}$ for which, computing a robust solution can be NP-hard in the absence of a regularization constraint. For an interval uncertainty set $\mathcal{U}$, problem $P(\mathcal{V}(\mathcal{U},\rho))$ is a linear semidefinite programming problem which can be solved efficiently. Note that $\Phi_\mu(\rho) = v^*$, where v^* is the optimal value of problem $P(\mathcal{V}(\mathcal{U},\rho))$.

Now we illustrate the effect of regularization on stability of the robust solution with an interval uncertainty using the portfolio execution described in Example 3.1. We use the same $M = 50$ perturbations $\Delta\bar{G}^{(\ell)}$ in (13), which are used in Figs. 1 and 2. Regularized robust solutions are computed in MATLAB 7.9 using (25) and (26). Problem (26) is solved using SEDUMI [49] through CVX, a package for specifying and solving convex programs [32] within MATLAB.

Figure 4 illustrates sensitivity of the actual robust efficient frontier corresponding to the regularized robust execution strategy to perturbation in the uncertainty set. The actual robust frontier for the regularized robust solutions is the worst case mean and variance with respect to the original uncertainty set $\mathcal{U}$. Comparing Fig. 4 with Fig. 1, we observe clear improvement in stability of the regularized robust solution. Furthermore, Fig. 4 indicates that increasing the regularization parameter ρ reduces variation in the actual robust frontiers.

Figure 5 illustrates stability of the regularized robust optimal execution strategy when $\mu = 0$ for two regularization parameter values ρ. Comparing the left plots with

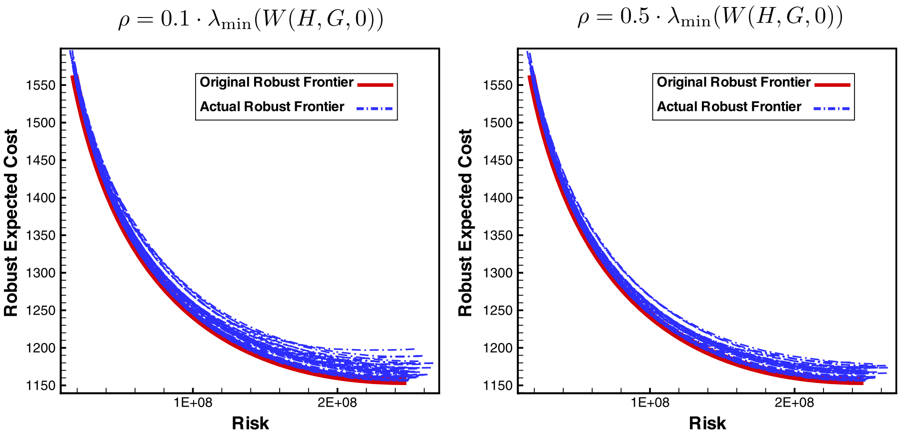

Fig. 4 Sensitivity of the robust efficient frontier for the regularized robust optimal execution strategy to 5 % perturbation in the upper bound of the uncertainty set $\mathcal{U}_G$. Regularization is applied to the three asset liquidation Example 3.1

the right plots in Fig. 5 indicates that the sensitivity is larger for a smaller regularization parameter ρ. In addition, the comparison between Figs. 5 and 2 indicates that the regularized robust optimal execution strategy has a more stable behavior to perturbation in the upper bound of the interval uncertainty set $\mathcal{U}_G$ than the classical robust optimal execution strategy. Note that, for both regularization parameter values, the worst case original permanent impact matrix G_u for $P(\mathcal{V}(\mathcal{U}, \rho))$ is symmetric; thus the regularized robust optimal execution strategy is the naive strategy, which follows from Proposition 2.1 in [42]. For a perturbed uncertainty set $\overline{\mathcal{U}} = \mathcal{U}_H \times \overline{\mathcal{U}}_G$, the worst case permanent impact matrix G_u from problem $P(\mathcal{V}(\overline{\mathcal{U}}, \rho))$ is typically not symmetric; thus the strategy can differ significantly from the naive strategy.

Next we formally analyze stability of the regularized robust solution.

6 Stability of the regularized robust optimal execution strategy

In this section we establish a bound on the change in the regularized robust optimal execution strategy, when the uncertainty set is perturbed. This bound explicitly indicates how the regularization parameter ρ affects sensitivity of the regularized robust solution to variation in the uncertainty set. In addition, we show that the change in the regularized robust solution converges to zero when the change in the uncertainty set $\mathcal{U}$ converges to zero.

We measure perturbation in the uncertainty set by the Hausdorff distance [35], which quantifies how far two subsets in a metric space are from each other. Given a metric space $(\mathcal{X}, d)$, the Hausdorff distance between two subsets $\mathcal{S}, \mathcal{T} \subseteq \mathcal{X}$ is defined by:

$$\mathbf{Haus}_d(\mathcal{S}, \mathcal{T}) \stackrel{\text{def}}{=} \max\left\{\sup_{s\in\mathcal{S}} \inf_{t\in\mathcal{T}} d(s,t), \sup_{t\in\mathcal{T}} \inf_{s\in\mathcal{S}} d(s,t)\right\},$$

see, e.g., Remark 4.40 of [14] for a more detailed discussion.

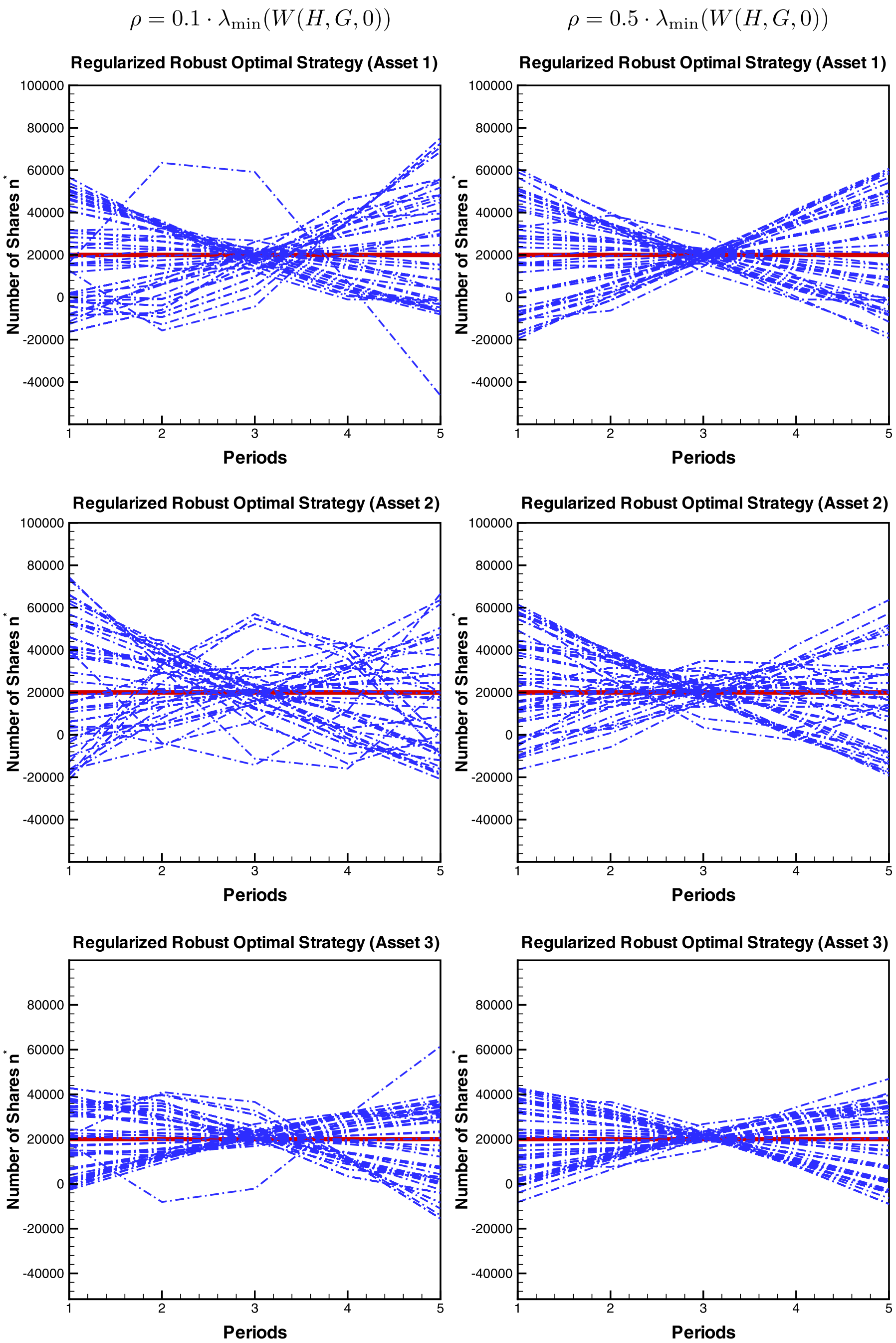

Fig. 5 Sensitivity of the regularized robust optimal execution strategy ($\mu = 0$) to perturbation in the upper bound of the uncertainty set $\mathcal{U}_G$, for Example 3.1

When both subsets $\mathcal{S}$ and $\mathcal{T}$ are bounded, $\mathbf{Haus}_d(\mathcal{S},\mathcal{T})$ is finite. The Hausdorff distance, defined on a metric space $(\mathcal{X},d)$, is a metric on the set of all *non-empty compact* subsets of $\mathcal{X}$, see, e.g., Proposition 4.1.8 of [44]. This metric has been previously used to measure perturbation to a set, see, e.g., [3]. Here, we define the Hausdorff metric induced by the metric d, below, on $\mathbb{R}^{2m^2}$:

$$d\big((H_1,G_1),(H_2,G_2)\big) \stackrel{\text{def}}{=} \frac{2}{\tau}\|H_1-H_2\|_2+\|G_1-G_2\|_2. \tag{35}$$

The norm $\|\cdot\|_2$ here denotes the matrix 2-norm.

Measuring perturbation in the uncertainty set using $\mathbf{Haus}_d$, we show next that, as $\mathbf{Haus}_d(\mathcal{U},\overline{\mathcal{U}})\to 0$, the distance of the regularized robust optimal strategies corresponding to $\mathcal{U}$ and $\overline{\mathcal{U}}$ also approaches zero. Our analysis mainly relies on results in [42] and [24]. Below, $\lambda_{\max}(\cdot)$ denotes the maximum eigenvalue.

Let $\{\mathcal{S}_k\}_k$ be a sequence of closed subsets of a compact set $\mathcal{T}$ in a metric space $(\mathcal{X},d)$. Theorem 2.1 in [24] implies that when $\{\mathcal{S}_k\}_k$ approaches a compact set $\mathcal{S}\subseteq\mathcal{X}$, i.e., $\mathbf{Haus}_d(\mathcal{S}_k,\mathcal{S})\to 0$, then any sequence of solutions of minimizing a continuous function f over $\mathcal{S}_k$ contains at least one convergent subsequence, and all cluster points are solutions of $\min_{x\in\mathcal{S}} f(x)$. A simplified version of this result is summarized in Theorem 6.1. Notice that Theorem 2.1 in [24] is more general than Theorem 6.1 presented here in the sense that it also allows a sequence of objective functions to uniformly converges to the objective function f.

Theorem 6.1 *Let $\mathcal{X}$ be a first-countable Hausdorff space and $\mathcal{T}$ is a sequentially compact nonempty subset of $\mathcal{X}$. Assume that $\mathcal{R}$ and $\mathcal{R}_k$ are closed subsets of $\mathcal{X}$ such that $\mathcal{R}\cap\mathcal{T}$ is nonempty, and $\mathcal{R}_k\cap\mathcal{T}\to\mathcal{R}\cap\mathcal{T}$. Let f be a real-valued function defined on $\mathcal{X}$ which is continuous on an open set containing $\mathcal{R}\cap\mathcal{T}$. Then for k large, there exists $x_k\in\mathcal{R}_k\cap\mathcal{T}$ such that $f(x_k)=\min_{x\in\mathcal{R}_k\cap\mathcal{T}} f(x)$. Furthermore, any (global) minimizing sequence $\{x_k\}$ of problems $\min_{x\in\mathcal{R}_k\cap\mathcal{T}} f(x)$ contains at least one convergent subsequence and all cluster points are (global) minimizing points of $f(x)$ in $\mathcal{R}\cap\mathcal{T}$.*

In Theorem 6.1, when the feasible sets $\mathcal{R}$ and $\mathcal{R}_k$ are compact, the following corollary can be derived:

Corollary 6.1 *Let $(\mathcal{X},d)$ be a metric space and $f:\mathcal{X}\to\mathbb{R}$ be continuous on $\mathcal{X}$. Consider the following problem:*

$$Q(\mathcal{R}):\max_{x\in\mathcal{R}} f(x),$$

where $\mathcal{R}\subseteq\mathcal{X}$ is nonempty and compact. Then for any sequence of nonempty compact subsets of $\mathcal{X}$, $\{\mathcal{R}_k\}_k$, with $\mathbf{Haus}_d(\mathcal{R},\mathcal{R}_k)\to 0$, and for any maximizing sequence $\{x_k\}_k$ of problems $Q(\mathcal{R}_k)$, there exists at least one convergent subsequence and all cluster points of $\{x_k\}_k$ are maximizing points of problem $Q(\mathcal{R})$.

Proof First note that every metric space is a first-countable Hausdorff space. Since $\mathcal{R}$ is compact, it is a bounded subset of the metric space $(\mathcal{X},d)$. Thus, it is contained in a ball of finite radius, i.e. there exists $x_0 \in \mathcal{X}$ and $M > 0$ such that $d(x_0, x) < M$, for all $x \in \mathcal{R}$.

Since $\mathbf{Haus}_d(\mathcal{R}, \mathcal{R}_k) \to 0$, there exists some K_0 such that for every $k \geq K_0$, $\mathbf{Haus}_d(\mathcal{R}, \mathcal{R}_k) \leq \epsilon_0$, and consequently $\sup_{x \in \mathcal{R}_k} \inf_{y \in \mathcal{R}} d(x, y) \leq \epsilon_0$. Therefore for every $x \in \mathcal{R}_k$, there exists some $y \in \mathcal{R}$ such that $d(x, y) \leq \epsilon_0$. Hence, $d(x, x_0) \leq d(x, y) + d(y, x_0) \leq \epsilon_0 + M$. Thus x is in the ball $B_{M+\epsilon_0}(x_0) = \{x \in \mathcal{X} : d(x, x_0) \leq \epsilon_0 + M\}$. Consequently, $\mathcal{R}_k \subseteq B_{M+\epsilon_0}(x_0)$, for every $k \geq K_0$. Denote the closure of $B_{M+\epsilon_0}(x_0)$ by $\mathcal{T}$. Thus $\mathcal{T}$ is a compact subset of $\mathcal{X}$ and $\mathcal{R}_k \subseteq \mathcal{T}$, for all $k \geq K_0$. The result then follows from Theorem 6.1 using the defined set $\mathcal{T}$. □

We precede stability analysis for the regularized robust strategy by the following auxiliary lemma, which is used in the proof for Theorem 6.2.

Lemma 6.1 *Let $\mathcal{R} = \mathcal{R}_c$ and a nonempty, convex, compact uncertainty set $\mathcal{U}$ be given. Assume that the regularization parameter $\rho > 0$ is chosen such that $\mathcal{V}(\mathcal{U}, \rho)$ is nonempty. Then problem (29), applied for the regularized uncertainty set $\mathcal{V}(\mathcal{U}, \rho)$, shares the same set of solutions with the following problem:*

$$\max \quad \frac{1}{\tau} \bar{S}^T \tilde{H} \bar{S} - \frac{1}{2} \big(b(\tilde{H}, \tilde{G}) + A^T \lambda\big)^T W^{-1}(\tilde{H}, \tilde{G}, \mu) \big(b(\tilde{H}, \tilde{G}) + A^T \lambda\big) - c^T \lambda \tag{36}$$

$$\text{s.t.} \quad \|\lambda\|_2 \leq \lambda_u$$

$$(\tilde{H}, \tilde{G}) \in \mathcal{V}(\mathcal{U}, \rho), \quad \lambda \in \mathbb{R}^r_+,$$

where the constant λ_u is given below:

$$\lambda_u = \frac{4\sqrt{m}\Lambda_u + 2\mu\tau\lambda_{\max}(C)}{(\rho + 2\mu\tau\lambda_{\min}(C)) \sin(\frac{\pi}{4N-2})} \left(1 + \frac{4\sqrt{m}\Lambda_u + 2\mu\tau\lambda_{\max}(C)}{2\sin^2(\frac{\pi}{4N-2})}\right) (\Lambda_u + 1) \|\bar{S}\|_2.$$

Here, $\Lambda_u = \max_{(\tilde{H}, \tilde{G}) \in \mathcal{U}} \|\tilde{\Theta}\|_2$ where $\tilde{\Theta}$ is the combined impact matrix corresponding to $\tilde{H}$ and $\tilde{G}$. Furthermore, the set of feasible points of problem (36) is compact.

Proof Recall that λ in (29) corresponds to the Lagrange dual multipliers for the inner minimization problem in (27), which is given below

$$\min_{z \in \mathcal{R}_c} \frac{1}{2} z^T W(\tilde{H}, \tilde{G}, \mu) z + b^T(\tilde{H}, \tilde{G}) z. \tag{37}$$

Let J be the set of indices of binding constraints in (7) defining $\mathcal{R}_c$ at the solution of problem (37). Lemma 3.1 in [42] yields

$$\lambda_{\min}\big(A_J A_J^T\big) \geq 4 \sin^2\left(\frac{\pi}{4N-2}\right). \tag{38}$$

Furthermore, using equation (42) in [42], we have

$$\big\| A_J^T \big\|_2 \leq 2. \tag{39}$$

Equation (3.15) in [33] implies that the Lagrange multiplier λ of the constraints defining $\mathcal{R}_c$ satisfies

$$\begin{aligned}\|\lambda\|_2 &\le \frac{2\lambda_{\max}(W(\tilde{H},\tilde{G},\mu))}{\lambda_{\min}(W(\tilde{H},\tilde{G},\mu))\cdot\sqrt{\lambda_{\min}(A_J A_J^T)}}\left(1+\frac{\lambda_{\max}(W(\tilde{H},\tilde{G},\mu))\cdot\|A_J^T\|_2}{\lambda_{\min}(A_J A_J^T)}\right)\\ &\quad\times\left(\|\tilde{\Theta}\bar{S}\|_2+\|\bar{S}\|_2\right)\\ &\le \frac{\lambda_{\max}(W(\tilde{H},\tilde{G},\mu))}{(\rho+2\mu\tau\lambda_{\min}(C))\cdot\sin(\frac{\pi}{4N-2})}\left(1+\frac{\lambda_{\max}(W(\tilde{H},\tilde{G},\mu))}{2\sin^2(\frac{\pi}{4N-2})}\right)\\ &\quad\times(\Lambda_u+1)\|\bar{S}\|_2, \end{aligned} \tag{40}$$

where inequalities (38), (39), and (33) are used to derive inequality (40). Notice that Λ_u is finite as $\mathcal{U}$ is compact.

For every $(\tilde{H},\tilde{G})\in\mathcal{V}(\mathcal{U},\rho)$, the matrix $W(\tilde{H},\tilde{G},0)$ is symmetric. Thus

$$\begin{aligned}\left\|W(\tilde{H},\tilde{G},0)\right\|_2 &\le \left\|W(\tilde{H},\tilde{G},0)\right\|_1\\ &\le \|\tilde{\Theta}\|_1+\left\|\tilde{\Theta}+\tilde{\Theta}^T\right\|_1+\left\|\tilde{\Theta}^T\right\|_1\\ &\le \|\tilde{\Theta}\|_1+\|\tilde{\Theta}\|_1+\left\|\tilde{\Theta}^T\right\|_1+\left\|\tilde{\Theta}^T\right\|_1\\ &= 2\|\tilde{\Theta}\|_1+2\|\tilde{\Theta}\|_\infty\le 4\sqrt{m}\Lambda_u. \end{aligned} \tag{41}$$

For the definition of $\|\cdot\|_\infty$ and $\|\cdot\|_1$, the reader is referred to Sect. 2.3.2 of [31].

Therefore,

$$\begin{aligned}\lambda_{\max}\big(W(\tilde{H},\tilde{G},\mu)\big) &\le \|W(\tilde{H},\tilde{G},0)\|_2+2\mu\tau\lambda_{\max}(C)\\ &\le 4\sqrt{m}\Lambda_u+2\mu\tau\lambda_{\max}(C). \end{aligned} \tag{42}$$

Using inequalities (41) and (42) in inequality (40) we get,

$$\begin{aligned}\|\lambda\|_2 &\le \frac{4\sqrt{m}\Lambda_u+2\mu\tau\lambda_{\max}(C)}{(\rho+2\mu\tau\lambda_{\min}(C))\cdot\sin(\frac{\pi}{4N-2})}\\ &\quad\times\left(1+\frac{4\sqrt{m}\Lambda_u+2\mu\tau\lambda_{\max}(C)}{2\sin^2(\frac{\pi}{4N-2})}\right)(\Lambda_u+1)\|\bar{S}\|_2. \end{aligned} \tag{43}$$

Thus optimal points of problem (29) satisfy inequality (43). Whence problems (29) and (36) have the same set of solutions.

The upper bound in inequality (43) depends on $\mathcal{U}$ and the constants ρ, N, and $\bar{S}$. Therefore, it is finite for any compact uncertainty set $\mathcal{U}\subseteq\mathbb{R}^{2m^2}$. Thus when $\mathcal{U}$ is nonempty and compact, the set of feasible points of problem (36) is closed and bounded, and consequently compact. □

We now estabilish the stability properties for the regularized robust optimal strategy.

Theorem 6.2 *Let the risk aversion parameter* $\mu \geq 0$ *and a nonempty convex compact uncertainty set* $\mathcal{U}$ *be given. Assume that the regularization parameter* $\rho > 0$ *is chosen such that* $\mathcal{V}(\mathcal{U}, \rho)$ *is nonempty. Denote a solution to the regularized robust problem* (34) *with respect to the uncertainty set* $\mathcal{U}$ *with* (z_u, H_u, G_u). *Let* $\overline{\mathcal{U}}$ *be any nonempty convex compact uncertainty set such that* $\mathcal{V}(\overline{\mathcal{U}}, \rho)$ *is nonempty, and* $(z_{\bar{u}}, H_{\bar{u}}, G_{\bar{u}})$ *be a solution to problem* (34) *with respect to* $\overline{\mathcal{U}}$. *Denote the combined impact matrices corresponding to* (H_u, G_u) *and* $(H_{\bar{u}}, G_{\bar{u}})$ *with* Θ_u *and* $\Theta_{\bar{u}}$, *respectively. Define* $\Lambda_u = \max_{(\tilde{H},\tilde{G})\in\mathcal{V}(\mathcal{U},\rho)} \|\tilde{\Theta}\|_2$. *Then the following hold*:

(a) *When the execution strategy is unconstrained, i.e.,* $\mathcal{R} = \mathcal{R}_0$,

$$\frac{\|z_u - z_{\bar{u}}\|_2}{\|\bar{S}\|_2} \leq \frac{1}{\beta_\rho}\left(1 + \frac{4\sqrt{m}}{\beta_\rho}\Lambda_u\right)\|\Theta_u - \Theta_{\bar{u}}\|_2, \tag{44}$$

where $\beta_\rho = \rho + 2\mu\tau\lambda_{\min}(C)$.

(b) *When buying is prohibited in the sell execution strategy, i.e.,* $\mathcal{R} = \mathcal{R}_c$, *there exists* $\varsigma_{u,\bar{u}} > 0$ *such that*

$$\frac{\|z_u - z_{\bar{u}}\|_2}{\|\bar{S}\|_2} \leq \varsigma_{u,\bar{u}}\big(1 + 4\sqrt{m}\,\varsigma_{u,\bar{u}}\big(\max\{1, \beta_u\} + \Lambda_u + \|\Theta_u - \Theta_{\bar{u}}\|_2\big)\big)\|\Theta_u - \Theta_{\bar{u}}\|_2, \tag{45}$$

where $\beta_u = 4\sqrt{m}\Lambda_u + 2\mu\tau\lambda_{\max}(C)$ *and*

$$\varsigma_{u,\bar{u}} \leq \frac{1}{\beta_\rho}\left(1 + \frac{\beta_u + 4\sqrt{m}\|\Theta_u - \Theta_{\bar{u}}\|_2}{\sin^2(\frac{\pi}{4N-2})\beta_\rho}\left(\frac{\beta_u + 4\sqrt{m}\|\Theta_u - \Theta_{\bar{u}}\|_2}{\beta_\rho} + 3\sin\left(\frac{\pi}{4N-2}\right)\right)\right). \tag{46}$$

(c) *In addition, for any uncertainty set* $\overline{\mathcal{U}}$ *with* $\mathbf{Haus}_d(\overline{\mathcal{U}}, \mathcal{U}) \to 0$, *we have* $\|z_u - z_{\bar{u}}\|_2 \to 0$, *when* $\mathcal{R}$ *equals either* $\mathcal{R}_c$ *or* $\mathcal{R}_0$, *and the metric* $d(\cdot,\cdot)$ *is defined in* (35).

Proof First we note that, since $\mathcal{U}$ and consequently $\mathcal{V}(\mathcal{U}, \rho)$ are compact, Λ_u is finite. Furthermore, since $(H_u, G_u) \in \mathcal{V}(\mathcal{U}, \rho)$ and $(H_{\bar{u}}, G_{\bar{u}}) \in \mathcal{V}(\overline{\mathcal{U}}, \rho)$, the matrices $W(H_u, G_u, \mu)$ and $W(H_{\bar{u}}, G_{\bar{u}}, \mu)$ are both positive definite. Whence, Proposition 4.1 implies that the corresponding regularized robust solutions are unique.

For notational simplicity, denote

$$W_u \stackrel{\text{def}}{=} W(H_u, G_u, \mu), \qquad W_{\bar{u}} \stackrel{\text{def}}{=} W(H_{\bar{u}}, G_{\bar{u}}, \mu).$$

Since $(H_u, G_u) \in \mathcal{V}(\mathcal{U}, \rho)$ and $(H_{\bar{u}}, G_{\bar{u}}) \in \mathcal{V}(\overline{\mathcal{U}}, \rho)$, inequality (33) yields

$$\min\big\{\lambda_{\min}(W_u), \lambda_{\min}(W_{\bar{u}})\big\} \geq \beta_\rho, \tag{47}$$

$$\hat{\lambda} \stackrel{\text{def}}{=} \max\big\{1, \lambda_{\max}(W_u)\big\} \geq \max\big\{1, \rho + 2\mu\tau\lambda_{\min}(C)\big\} \geq \beta_\rho. \tag{48}$$

Since the regularized robust solutions z_u and $z_{\bar{u}}$ solve the nominal optimal portfolio execution problem (5) with the impact matrices (H_u, G_u) and $(H_{\bar{u}}, G_{\bar{u}})$, respectively, Theorem 3.1 in [42] yields

$$\|z_u - z_{\bar{u}}\|_2 \le \frac{\|\bar{S}\|_2}{\min\{\lambda_{\min}(W_u), \lambda_{\min}(W_{\bar{u}})\}}\left(1 + \frac{4\sqrt{m}}{\min\{\lambda_{\min}(W_u), \lambda_{\min}(W_{\bar{u}})\}}\|\Theta_u\|_2\right) \times \|\Theta_u - \Theta_{\bar{u}}\|_2, \tag{49}$$

when $\mathcal{R} = \mathcal{R}_0$.

Applying inequality (47) in (49), and the fact that $\|\Theta_u\|_2 \le \Lambda_u$, we obtain inequality (44) and the proof of part (a) is completed.

Next, let $\mathcal{R} = \mathcal{R}_c$. Theorem 3.3 in [42] implies that

$$\|z_u - z_{\bar{u}}\|_2 \le \varsigma_{u,\bar{u}}\|\bar{S}\|_2\big(1 + 4\varsigma_{u,\bar{u}}\sqrt{m}\big(\hat{\lambda} + \|\Theta_u\|_2 + \|\Theta_u - \Theta_{\bar{u}}\|_2\big)\big)\|\Theta_u - \Theta_{\bar{u}}\|_2, \tag{50}$$

where

$$\varsigma_{u,\bar{u}} \le \frac{1}{\underline{\lambda}}\left(1 + \frac{(\bar{\lambda} + \underline{\lambda})}{2\sin^2(\frac{\pi}{4N-2})\hat{\lambda}}\left(\frac{\bar{\lambda}}{\hat{\lambda}} + 3\sin\left(\frac{\pi}{4N-2}\right)\right)\right), \tag{51}$$

with $\bar{\lambda} = \max_{\eta\in[0,1]} \lambda_{\max}(W_u + \eta(W_{\bar{u}} - W_u))$, $\underline{\lambda} = \min_{\eta\in[0,1]} \lambda_{\min}(W_u + \eta(W_{\bar{u}} - W_u))$, and $\hat{\lambda}$ is as in (48).

The Courant-Fischer Theorem yields

$$\begin{aligned}\underline{\lambda} &= \min_{\eta\in[0,1]} \lambda_{\min}\big(W_u + \eta(W_{\bar{u}} - W_u)\big) \\ &\ge \min_{\eta\in[0,1]} \big(\eta\lambda_{\min}(W_{\bar{u}}) + (1-\eta)\lambda_{\min}(W_u)\big) \ge \beta_\rho,\end{aligned} \tag{52}$$

where the last inequality comes from inequality (33).

Since the matrix $W_u - W_{\bar{u}}$ is symmetric, we have $\|W_u - W_{\bar{u}}\|_1 = \|W_u - W_{\bar{u}}\|_\infty$. Hence, Corollary 2.3.2 in [31] yields

$$\|W_u - W_{\bar{u}}\|_2 \le \sqrt{\|W_u - W_{\bar{u}}\|_1\|W_u - W_{\bar{u}}\|_\infty} = \|W_u - W_{\bar{u}}\|_1.$$

Therefore we have

$$\begin{aligned}\|W_u - W_{\bar{u}}\|_2 &\le \|W_u - W_{\bar{u}}\|_1 \\ &\le \|\Theta_u - \Theta_{\bar{u}}\|_1 + \left\|\Theta_u - \Theta_{\bar{u}} + (\Theta_u - \Theta_{\bar{u}})^T\right\|_1 + \left\|(\Theta_u - \Theta_{\bar{u}})^T\right\|_1 \\ &\le 2\|\Theta_u - \Theta_{\bar{u}}\|_1 + 2\left\|(\Theta_u - \Theta_{\bar{u}})^T\right\|_1 \\ &= 2\|\Theta_u - \Theta_{\bar{u}}\|_1 + 2\|\Theta_u - \Theta_{\bar{u}}\|_\infty \\ &\le 4\sqrt{m}\|\Theta_u - \Theta_{\bar{u}}\|_2.\end{aligned} \tag{53}$$

This result along with the Courant-Fischer Theorem imply that

$$\begin{aligned}\bar{\lambda} &= \max_{\eta\in[0,1]} \lambda_{\max}\big(W_u + \eta(W_{\bar{u}} - W_u)\big)\\ &\le \max_{\eta\in[0,1]} \big(\lambda_{\max}(W_u) + \eta\lambda_{\max}(W_{\bar{u}} - W_u)\big)\\ &\le \lambda_{\max}(W_u) + \max_{\eta\in[0,1]} \eta\|W_{\bar{u}} - W_u\|_2\\ &\le \lambda_{\max}(W_u) + \|W_{\bar{u}} - W_u\|_2\\ &\le \lambda_{\max}(W_u) + 4\sqrt{m}\|\Theta_{\bar{u}} - \Theta_u\|_2. \end{aligned} \tag{54}$$

Since $(H_u, G_u) \in \mathcal{U}$, inequality (42) yields $\lambda_{\max}(W_u) \le \beta_u$. Using this inequality in inequality (54), we get

$$\begin{aligned}&\bar{\lambda} \le \beta_u + 4\sqrt{m}\|\Theta_{\bar{u}} - \Theta_u\|_2,\\ &\bar{\lambda} + \underline{\lambda} \le 2\bar{\lambda} \le 2\big(\beta_u + 4\sqrt{m}\|\Theta_{\bar{u}} - \Theta_u\|_2\big).\end{aligned}$$

Applying these inequalities, along with inequalities (48) and (52), in (51) yields inequality (46). Furthermore, using inequalities $\|\Theta_u\|_2 \le \Lambda_u$ and $\|W_u\|_2 = \lambda_{\max}(W_u) \le \beta_u$ in (50), inequality (45) is obtained. This completes the proof of part (b).

The proof of part (c) relies on Theorem 6.1, established in [24], for problems (31) and (36). Since the matrix $W(\tilde{H}, \tilde{G}, \mu)$ is positive definite over $\mathcal{V}(\mathcal{U}, \rho)$, the entries of the inverse matrix $W^{-1}(\tilde{H}, \tilde{G}, \mu)$ are continuous functions of the entries of the matrices $\tilde{H}$ and $\tilde{G}$ (see, e.g., [4]). Whence, the objective functions of problems (31) and (36) are continuous with respect to elements of $\tilde{H}$, $\tilde{G}$, and λ.

First consider the case when the set of feasible execution strategies is $\mathcal{R}_0$. Suppose $\mathbf{Haus}_d(\bar{\mathcal{U}}, \mathcal{U}) \to 0$ and $\|z_u - z_{\bar{u}}\|_2 \not\to 0$. Thus there exists some $\epsilon > 0$ such that for every k there exists some $\overline{\mathcal{U}}_k \subseteq \mathbb{R}^{2m^2}$ with $\mathbf{Haus}_d(\overline{\mathcal{U}}_k, \mathcal{U}) < \frac{1}{k}$ and $\|z_u - z_{\bar{u}_k}\|_2 > \epsilon$. Here $z_{\bar{u}_k}$ is the regularized robust solution corresponding to the uncertainty set $\overline{\mathcal{U}}_k$. Let $\{(H_{\bar{u}_k}, G_{\bar{u}_k})\}_k$ be a sequence of solutions of problem (31) with the uncertainty sets $\{\overline{\mathcal{U}}_k\}$. Corollary 6.1 yields that there exists a subsequence $\{(H_{\bar{u}_{k_i}}, G_{\bar{u}_{k_i}})\}_i$ of the sequence $\{(H_{\bar{u}_k}, G_{\bar{u}_k})\}_k$ that approaches to a solution (H_u, G_u) of problem (31) with the uncertainty set $\mathcal{U}$. Thus, for i sufficiently large, $d((H_u, G_u), (H_{\bar{u}_{k_i}}, G_{\bar{u}_{k_i}})) \to 0$. Consequently, $\|\Theta_u - \Theta_{\bar{u}_{k_i}}\|_2 \to 0$, because $\|\Theta_u - \Theta_{\bar{u}_{k_i}}\|_2 \le d((H_u, G_u), (H_{\bar{u}_{k_i}}, G_{\bar{u}_{k_i}}))$. Using inequality (44) and the fact that the regularized robust solutions z_u and $z_{\bar{u}_{k_i}}$ are unique, we get $\|z_u - z_{\bar{u}_{k_i}}\|_2 \to 0$, for i large enough. This result is in contradiction to $\|z_u - z_{\bar{u}_{k_i}}\|_2 > \epsilon$. Whence, $\|z_u - z_{\bar{u}}\|_2 \to 0$ as $\mathbf{Haus}_d(\bar{\mathcal{U}}, \mathcal{U}) \to 0$.

Now, let $\mathcal{R} = \mathcal{R}_c$. Recall that (H_u, G_u) solves problem (24) or equivalently problem (29). Furthermore, Lemma 6.1 indicates that (H_u, G_u) constitutes a solution of problem (36) in which the set of feasible points is compact. Therefore, Corollary 6.1 is applicable to problem (36). A similar discussion, as in the previous case, through Corollary 6.1 and inequalities (45) and (46) completes the proof of part (c), when $\mathcal{R} = \mathcal{R}_c$. □

Theorem 6.2 implies that small variations in the uncertainty set $\mathcal{U}$ result in small changes in the regularized robust solution. In other words, the regularized robust solution is asymptotically stable with respect to change in the uncertainty set.

7 Implications of regularization

In this section, we discuss additional implications of the proposed regularization on the robust solution, the robust optimal value, and the efficient frontier.

7.1 Implications on the optimal execution strategy

Here, we analyze how the regularization parameter affects some characteristics of the regularized robust optimal execution strategy.

For every $(\tilde{H}, \tilde{G}) \in \mathcal{V}(\mathcal{U}, \rho)$, inequality (33) yields

$$\left\| W^{-1}(\tilde{H}, \tilde{G}, \mu)\right\|_2 = \lambda_{\max}\left(W^{-1}(\tilde{H}, \tilde{G}, \mu)\right) = \frac{1}{\lambda_{\min}(W(\tilde{H}, \tilde{G}, \mu))} \leq \frac{1}{\rho + 2\mu\tau\lambda_{\min}(C)}. \tag{55}$$

The following proposition shows that the regularized robust solution satisfies a Tikhonov-type regularization constraint, when $\mathcal{R} = \mathcal{R}_0$.

Proposition 7.1 *Let $\mathcal{R} = \mathcal{R}_0$, the risk aversion parameter $\mu \geq 0$, and the nonempty convex compact uncertainty set $\mathcal{U}$ be given. Assume that the regularization parameter ρ is chosen such that $\mathcal{V}(\mathcal{U}, \rho)$ is nonempty. Then the regularized robust solution $z_u \in \mathcal{R}_0$ of problem* (34) *satisfies*:

$$\frac{\|z_u\|_2}{\|\bar{S}\|_2} \leq \frac{\Lambda_u}{\rho + 2\mu\tau\lambda_{\min}(C)}, \tag{56}$$

where $\Lambda_u = \max_{(\tilde{H}, \tilde{G}) \in \mathcal{V}(\mathcal{U}, \rho)} \|\Theta_u\|_2$.

Proof When the set of feasible execution strategies is given by $\mathcal{R}_0$, the regularized robust solution is determined by equation (23). Applying inequality (55), we get:

$$\|z_u\|_2 = \left\| -W^{-1}(H_u, G_u, \mu) b(H_u, G_u)\right\|_2 \leq \left\| W^{-1}(H_u, G_u, \mu)\right\|_2 \left\| b(H_u, G_u)\right\|_2 \leq \frac{\|b(H_u, G_u)\|_2}{\rho + 2\mu\tau\lambda_{\min}(C)}.$$

Using $\|b(H_u, G_u)\|_2 = \| - \Theta_u \bar{S}\|_2 \leq \|\Theta_u\|_2 \|\bar{S}\|_2 \leq \Lambda_u \|\bar{S}\|_2$ in the above inequality completes the proof of inequality (56). □

Proposition 7.1 indicates that including the regularization constraint in the uncertainty set implicitly offers a solution that satisfies a two-norm constraint on the execution strategy, a form of the Tikhonov regularization.

In addition, the regularization parameter also affects the Euclidean distance between the regularized robust optimal execution strategy and the naive strategy. The naive strategy can be used as a benchmark since it is always the solution for the nominal optimal portfolio execution problem (5), when $\mu = 0$, the permanent impact matrix G is symmetric, and Θ is positive definite (see, e.g., Proposition 2.1 in [42]). Furthermore, when the permanent impact matrix $\tilde{G}$ is symmetric for every element in $\mathcal{U}$, which holds in a single asset case, the robust optimal execution strategy is the naive strategy, regardless of the choice of the uncertainty set. A bound on the distance between the regularized robust optimal execution strategy and the naive strategy is established in the next proposition.

Proposition 7.2 *Let $\mathcal{R} = \mathcal{R}_0$, the risk aversion parameter $\mu \geq 0$, and the nonempty convex compact uncertainty set $\mathcal{U}$ be given. Assume the regularization parameter ρ is chosen such that $\mathcal{V}(\mathcal{U}, \rho)$ is nonempty. Then, the regularized robust optimal execution strategy z_u of problem (34) satisfies:*

$$\frac{\|z_u - z_n\|_2}{\|\bar{S}\|_2} \leq \frac{\Lambda_u}{4\sin^2(\frac{\pi}{2N})\rho}\left(1 + \frac{4\sqrt{m}\Lambda_u + 2\mu\tau\lambda_{\max}(C)}{\beta_\rho}\right), \tag{57}$$

where $\beta_\rho = \rho + 2\mu\tau\lambda_{\min}(C)$ and z_n represents the naive strategy $x_k = (\frac{N-k}{N})\bar{S}$ for $k = 1, 2, \ldots, N$.

Proof Let $(H_u, G_u) \in \mathcal{V}(\mathcal{U}, \rho)$ be a solution of problem (26) with the uncertainty set $\mathcal{V}(\mathcal{U}, \rho)$. The unique regularized robust solution from problem (34) is then $z_u = -W(H_u, G_u, \mu)^{-1} b(H_u, G_u)$. For simplicity, denote

$$W_u \stackrel{\text{def}}{=} W(H_u, G_u, \mu), \qquad b_u \stackrel{\text{def}}{=} b(H_u, G_u).$$

Notice that $\|b_u\|_2 \leq \|\Theta_u\|_2\|\bar{S}\|_2 \leq \Lambda_u\|\bar{S}\|_2$. Whence

$$\left\|b\left(H_u^T, G_u^T\right)\right\|_2 = \left\|\Theta_u^T\bar{S}\right\|_2 \leq \left\|\Theta_u^T\right\|_2\|\bar{S}\|_2 = \|\Theta_u\|_2\|\bar{S}\|_2 \leq \Lambda_u\|\bar{S}\|_2. \tag{58}$$

Since the naive strategy minimizes the expected execution cost for any symmetric permanent impact matrix where the combined impact matrix is positive definite (see, e.g., Proposition 2.1 in [42]), it is the unique optimal execution strategy corresponding to the impact matrices $H_u + H_u^T$ and $G_u + G_u^T$, i.e., $z_n = -W_n^{-1} b_n$, where

$$W_n \stackrel{\text{def}}{=} W\left(H_u + H_u^T, G_u + G_u^T, 0\right), \qquad b_n \stackrel{\text{def}}{=} b\left(H_u + H_u^T, G_u + G_u^T\right).$$

Therefore

$$\begin{aligned} W_n(z_u - z_n) &= W_n\left(-W_u^{-1}b_u - \left(-W_n^{-1}b_n\right)\right) = -W_n W_u^{-1} b_u + b_n \\ &= (W_u - W_n)W_u^{-1}b_u - b_u + b_n. \end{aligned}$$

Using $b_n - b_u = b(H_u^T, G_u^T)$ and $W_u - W_n = W(-H_u^T, -G_u^T, \mu)$, we have:

$$
\begin{aligned}
&\|z_u - z_n\|_2 \\
&\quad = \big\| W_n^{-1}\big((W_u - W_n)W_u^{-1}b_u - b_u + b_n\big)\big\|_2 \\
&\quad \le \big\| W_n^{-1}\big\|_2\big(\| W_u - W_n\|_2\big\| W_u^{-1}\big\|_2\|b_u\|_2 + \|-b_u + b_n\|_2\big) \\
&\quad = \big\| W_n^{-1}\big\|_2\big(\big\| W\big(-H_u^T, -G_u^T, \mu\big)\big\|_2\| W_u^{-1}\|_2\|b_u\|_2 + \big\|b\big(H_u^T, G_u^T\big)\big\|_2\big) \\
&\quad = \frac{1}{\lambda_{\min}(W_n)}\left(\big\| W\big(-H_u^T, -G_u^T, \mu\big)\big\|_2\frac{\|b_u\|_2}{\lambda_{\min}(W_u)} + \big\|b\big(H_u^T, G_u^T\big)\big\|_2\right) \\
&\quad \le \frac{1}{\lambda_{\min}(W_n)}\left(\big\| W\big(-H_u^T, -G_u^T, \mu\big)\big\|_2\frac{\Lambda_u\|\bar{S}\|_2}{\rho + 2\mu\tau\lambda_{\min}(C)} + \Lambda_u\|\bar{S}\|_2\right) \qquad (59) \\
&\quad \le \frac{1}{\lambda_{\min}(W_n)}\left(\big(\big\| W\big(-H_u^T, -G_u^T, 0\big)\big\|_2 + 2\mu\tau\lambda_{\max}(C)\big)\frac{\Lambda_u\|\bar{S}\|_2}{\beta_\rho} + \Lambda_u\|\bar{S}\|_2\right). \qquad (60)
\end{aligned}
$$

where inequality (59) comes from (33) and (58). Inequality (60) comes from the Courant-Fischer theorem.

Corollary 2.1. in [42] applied to W_n implies that

$$\lambda_{\min}(W_n) = 4\sin^2\left(\frac{\pi}{2N}\right)\lambda_{\min}\big(\Theta_u + \Theta_u^T\big).$$

Since $(H_u, G_u) \in \mathcal{V}(\mathcal{U}, \rho)$, we have $\lambda_{\min}(W(H_u, G_u, 0)) \ge \rho$. This yields $\lambda_{\min}(\Theta_u + \Theta_u^T) \ge \rho$, as the matrix $\Theta_u + \Theta_u^T$ is a leading principle submatrix of $\lambda_{\min}(W(H_u, G_u, 0))$. Thus we get

$$\lambda_{\min}(W_n) \ge 4\sin^2\left(\frac{\pi}{2N}\right)\rho. \qquad (61)$$

Furthermore, a vector $(a_1^T, a_2^T, \dots, a_{N-1}^T)^T$ is an eigenvector of $W(H_u, G_u, 0)$ associated with the eigenvalue λ if and only if the vector $(-a_{N-1}^T, -a_{N-2}^T, \dots, -a_1^T)^T$ is an eigenvector of $W(-H_u^T, -G_u^T, 0)$ for the same eigenvalue. Thus,

$$\big\| W\big(-H_u^T, -G_u^T, 0\big)\big\|_2 = \big\| W(H_u, G_u, 0)\big\|_2 \le 4\sqrt{m}\Lambda_u, \qquad (62)$$

where the inequality comes from (41). Using inequalities (61) and (62) in (60) completes the proof of (57). □

Table 1 computationally illustrates this property for liquidating the three assets in Example 3.1. In this example, the Euclidean distance between the naive strategy and the regularized robust solution decreases as the regularization parameter increases. Figure 6 further illustrates the impact of the regularization parameter ρ on the regularized robust optimal execution strategy. We observe that, as the regularization parameter increases, trading for the first and second assets becomes more even while trading for the third asset becomes slightly more uneven. Plots in Fig. 6 further depict the difference between the regularized robust optimal execution strategies (for $\rho_0 = 0.8, 1, 1.3$) and the (classical) robust solution.

Table 1 Difference between the regularized robust solution and the naive strategy, for liquidating the three assets in Example 3.1. Here the regularization parameter equals $\rho = \rho_0 \cdot \lambda_{\min}(W(H, G, 0))$

ρ_0	$\|z_u - z_n\|_2/\|\bar{S}\|_2$	
	$\mu = 0.5 \times 10^{-6}$	$\mu = 0.5 \times 10^{-7}$
0.1	0.587858	0.084620
0.8	0.501393	0.070671
1	0.492655	0.070255
1.3	0.488592	0.070077

Proposition 7.2 indicates that if ρ increases, the upper bound on the difference between the regularized robust optimal execution strategy and the naive strategy decreases. This property demonstrates a difference between the regularization parameter ρ and the risk aversion parameter μ. When a large risk aversion parameter μ is chosen, the optimal execution strategy becomes close to the strategy of liquidating the entire holding in the first period. We note that Proposition 7.2 can be extended for more general feasible sets $\mathcal{R}$.

7.2 Implications on the efficient frontier

A mean-variance efficient frontier clearly depicts the performance of a strategy in terms of the cost and risk. Here, we study impact of regularization on the efficient frontier. Under the assumed model, following (4), variance of the execution cost does not depend on the impact matrices. Whence, robust optimization problem $RC(\mathcal{U})$ minimizes the weighted sum of the worst case mean of the execution cost and the variance of the execution cost.

Firstly we compare the nominal mean-variance performance of the nominal optimal execution strategy with that of the regularized robust optimal execution strategy. Every robust solution is a feasible point for the nominal optimal portfolio execution problem with nominal impact matrices. Thus, the nominal efficient frontier of nominal solutions is always below the nominal efficient frontier of the robust solution with respect to any uncertainty set $\mathcal{U}$. The nominal frontier of the robust solution is the curve of the nominal mean and variance points corresponding to the robust optimal execution strategy.

We consider here a single asset execution example to illustrate. Left plot in Fig. 7 compares the nominal frontier of the nominal solution with the nominal frontier of the regularized robust solution. At the left end of the frontiers (corresponding to $\mu \to 10^{-3}$), all of the nominal frontiers converge to a single point, which corresponds to the optimal execution strategy of minimizing the variance of the execution cost, $n_1^* = \bar{S}$ and $n_i^* = 0$ for $i = 2, \ldots, N$. As μ increases, more weight is given to minimizing the expected cost and the difference among the frontiers becomes more prominent. The difference increases as the regularization parameter increases. Since here the nominal permanent impact matrices and worst-case permanent impact matrix are symmetric, the naive strategy is optimal for the nominal and robust problems, when $\mu = 0$. Hence, the frontiers also converge to a single point at the right end. An interesting observation from the left plot in Fig. 7, is that the nominal frontier of the regularized robust solution does not intersect with the nominal frontier of the nominal solution, except at its ends (comparing it with Fig. 1 in [56]). This suggests that

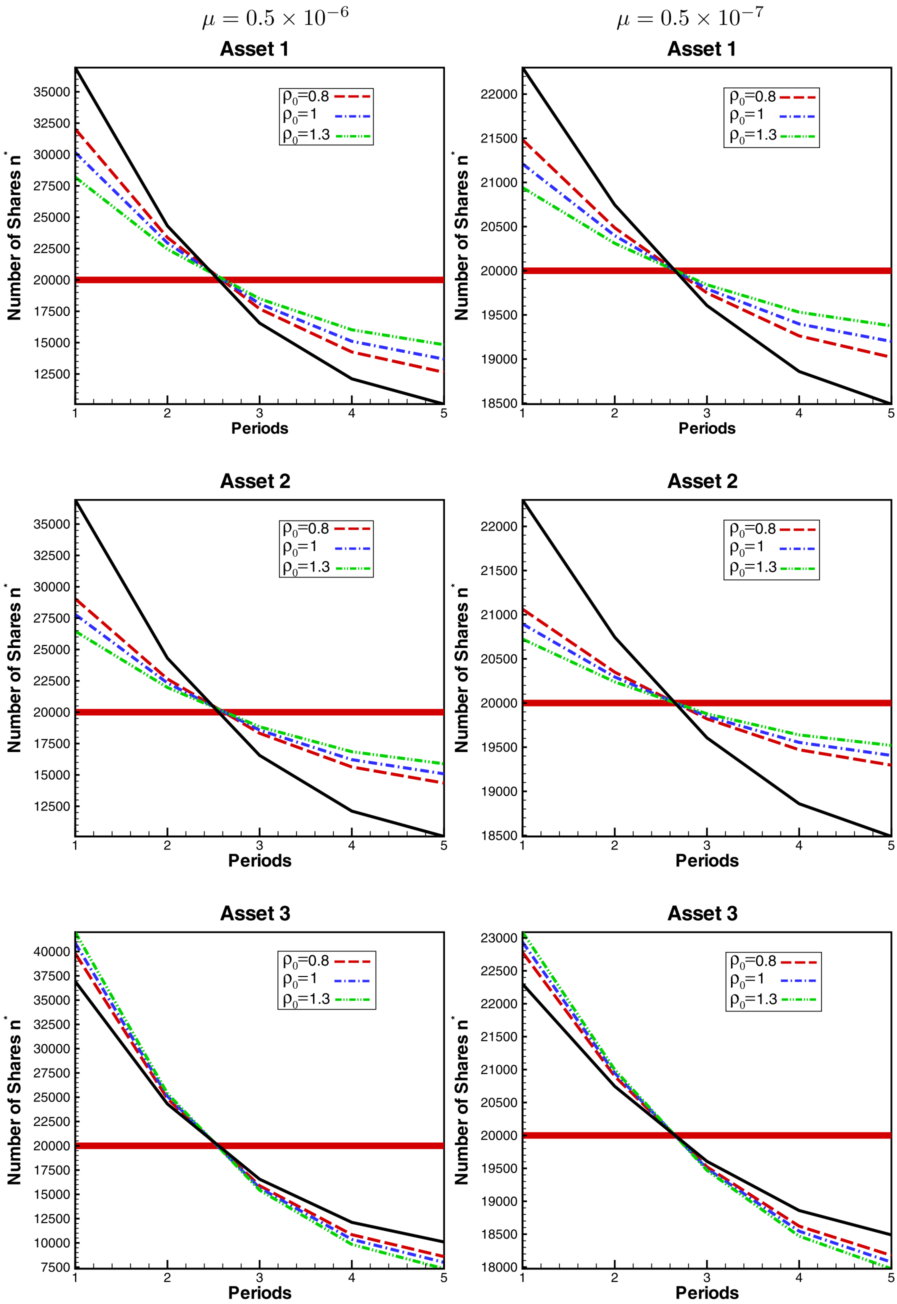

Fig. 6 Execution strategies for liquidating three assets in Example 3.1 with the uncertainty set as in (12) and $\rho = \rho_0 \cdot \lambda_{\min}(W(H, G, 0))$. The feasible set is $\mathcal{R} = \mathcal{R}_0$. The *thick solid line* depicts the naive strategy. The *thin solid line* represents the classical (unregularized) robust solution

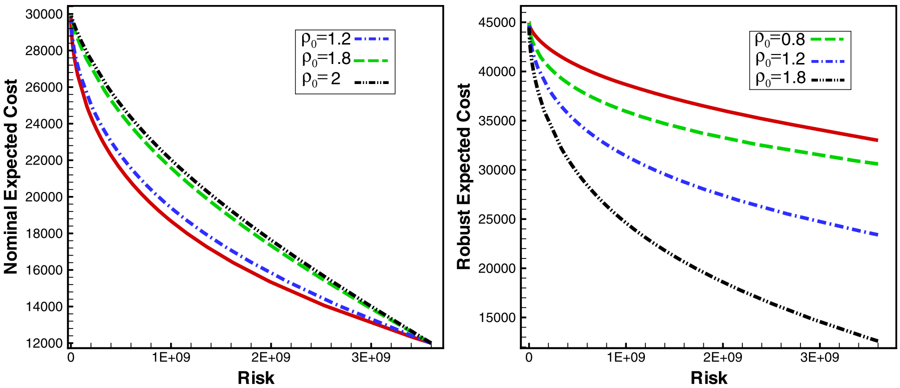

Fig. 7 A single asset trading with $C = 0.003$, $H = 10^{-5} \cdot C$, $G = 0.5 \times 10^{-5} \cdot C$. The uncertainty set is $\mathcal{U} = \mathcal{U}_H \times \mathcal{U}_G$, where $\mathcal{U}_H = [0.5 \cdot H,\ 1.5 \cdot H]$ and $\mathcal{U}_G = [0.5 \cdot G,\ 4 \cdot G]$. Frontiers are for $\mu \in [0, 10^{-3}]$ and the feasible set of execution strategies is $\mathcal{R}_0$. The regularization parameter equals $\rho = \rho_0 \cdot \lambda_{\min}(W(H, G, 0))$. *Right plot* illustrates robust frontier (with respect to $\mathcal{U}$) of nominal solutions (depicted by *solid line*), and robust frontiers (with respect to $\mathcal{V}(\mathcal{U}, \rho)$) of regularized solutions for several choices of ρ. *Left plot* illustrates the nominal efficient frontier of nominal solution (depicted by *solid line*) and nominal frontier of regularized robust solutions

a regularized robust solution cannot be obtained simply by adjusting μ in the nominal optimization framework. Hence, in general (for general uncertainty sets) there is no correspondence between the risk aversion parameter μ and the regularization parameter ρ.

Next we assess robust performance by examining the robust frontier. The robust frontier (with respect to an uncertainty set $\mathcal{U}$) of the nominal solution is the worst case mean and variance of the nominal solution. Since the solution of the nominal optimal portfolio execution problem is feasible for problem $RC(\mathcal{U})$, the variance and worst case mean of its corresponding execution cost are no smaller than those of the robust solution. Therefore, the robust frontier of the nominal optimal execution strategy is always above the robust efficient frontier of the robust optimal execution strategy. This has also been computationally observed in [55] for a single period traditional portfolio optimization, when only mean return is subject to uncertainty.

Conservativeness of the regularized robust optimal execution strategy can be adjusted through the regularization parameter. As the regularization parameter ρ increases, the size of the regularized uncertainty set decreases. Hence, $\Phi_\mu(\rho_1) \geq \Phi_\mu(\rho_2)$, when $\rho_1 \leq \rho_2$. Here, $\Phi_\mu(\cdot)$ is the robust optimal value defined in (34). Hence, the regularized robust solution becomes less conservative. In particular, $\Phi_\mu(\rho) \leq \Phi_\mu(0)$, for every $\rho \geq 0$.

Let the feasible region $\mathcal{R}$ be closed and convex, and the uncertainty set $\mathcal{U}$ be nonempty, convex, and compact. Assume ρ_1 and ρ_2 are two regularization parameters where $0 \leq \rho_1 \leq \rho_2$, and the sets $\mathcal{V}(\mathcal{U}, \rho_1)$ and $\mathcal{V}(\mathcal{U}, \rho_2)$ are nonempty. Then the (mean-variance) robust frontier with respect to $\mathcal{V}(\mathcal{U}, \rho_2)$ of the regularized robust solutions corresponding to ρ_2 is always below the mean-variance robust frontier with respect to $\mathcal{V}(\mathcal{U}, \rho_1)$ of the regularized robust solution corresponding to ρ_1. The property is depicted in the right plot in Fig. 7.

In regularized robust optimization, when the regularization parameter ρ increases, the robust frontier with respect to the regularized uncertainty set of the regularized robust solution is pushed down. The following discussion illustrates this result.

Let the regularized robust optimal execution strategy, corresponding to the risk aversion parameter μ and the regularized uncertainty set $\mathcal{V}(\mathcal{U}, \rho_1)$, be z_{ρ_1}. Denote the variance and robust mean of the execution cost corresponding to z_{ρ_1} with $\mathbf{V}_1$ and $\mathbf{E}_1$, respectively:

$$\mathbf{V}_1 \stackrel{\text{def}}{=} \tau z_{\rho_1}^T (I \otimes C) z_{\rho_1},$$

$$\mathbf{E}_1 \stackrel{\text{def}}{=} \max_{(\tilde{H},\tilde{G})\in\mathcal{V}(\mathcal{U},\rho_1)} \frac{1}{\tau}\bar{S}^T \tilde{H}\bar{S} + \frac{1}{2} z_{\rho_1}^T W(\tilde{H}, \tilde{G}, 0) z_{\rho_1} + b^T(\tilde{H}, \tilde{G}) z_{\rho_1}.$$

Let z_{ρ_2} be the regularized robust optimal execution strategy obtained from the regularized uncertainty set $\mathcal{V}(\mathcal{U}, \rho_2)$ and the risk aversion parameter $\hat{\mu}$ such that $\tau z_{\rho_2}^T (I \otimes C) z_{\rho_2} = \mathbf{V}_1$. Denote the robust expected execution cost corresponding to z_{ρ_2} with $\mathbf{E}_2$, i.e.,

$$\mathbf{E}_2 \stackrel{\text{def}}{=} \max_{(\tilde{H},\tilde{G})\in\mathcal{V}(\mathcal{U},\rho_2)} \frac{1}{\tau}\bar{S}^T \tilde{H}\bar{S} + \frac{1}{2} z_{\rho_2}^T W(\tilde{H}, \tilde{G}, 0) z_{\rho_2} + b^T(\tilde{H}, \tilde{G}) z_{\rho_2}.$$

Using Sion's convex-concave minimax theorem (see, e.g., Theorem 3 in [48]) and the fact that both $\mathcal{V}(\mathcal{U}, \rho_2)$ and $\mathcal{R}$ are convex, and the uncertainty set $\mathcal{V}(\mathcal{U}, \rho_2)$ is compact, we have:

$$\begin{aligned}
\mathbf{E}_2 + \hat{\mu}\,\mathbf{V}_1 &= \Phi_{\hat{\mu}}(\rho_2) \\
&= \max_{(\tilde{H},\tilde{G})\in\mathcal{V}(\mathcal{U},\rho_2)} \min_{z\in\mathcal{R}} \frac{1}{\tau}\bar{S}^T \tilde{H}\bar{S} + \frac{1}{2} z^T W(\tilde{H}, \tilde{G}, \hat{\mu}) z + b^T(\tilde{H}, \tilde{G}) z \qquad (63) \\
&\le \max_{(\tilde{H},\tilde{G})\in\mathcal{V}(\mathcal{U},\rho_2)} \frac{1}{\tau}\bar{S}^T \tilde{H}\bar{S} + \frac{1}{2} z_{\rho_1}^T W(\tilde{H}, \tilde{G}, \hat{\mu}) z_{\rho_1} + b^T(\tilde{H}, \tilde{G}) z_{\rho_1} \\
&\le \max_{(\tilde{H},\tilde{G})\in\mathcal{V}(\mathcal{U},\rho_1)} \frac{1}{\tau}\bar{S}^T \tilde{H}\bar{S} + \frac{1}{2} z_{\rho_1}^T W(\tilde{H}, \tilde{G}, \hat{\mu}) z_{\rho_1} + b^T(\tilde{H}, \tilde{G}) z_{\rho_1} \qquad (64) \\
&= \mathbf{E}_1 + \hat{\mu}\,\mathbf{V}_1,
\end{aligned}$$

where inequality (64) comes from the assumption $\rho_1 \le \rho_2$, which yields $\mathcal{V}(\mathcal{U}, \rho_2) \subseteq \mathcal{V}(\mathcal{U}, \rho_1)$.

Thus we obtained $\mathbf{E}_2 + \hat{\mu}\,\mathbf{V}_1 \le \mathbf{E}_1 + \hat{\mu}\,\mathbf{V}_1$ and consequently $\mathbf{E}_2 \le \mathbf{E}_1$. In other words, the point $(\mathbf{V}_1, \mathbf{E}_2)$ on the robust frontier corresponding to $\mathcal{V}(\mathcal{U}, \rho_2)$ is below the point $(\mathbf{V}_1, \mathbf{E}_1)$. Hence the robust frontier of $\mathcal{V}(\mathcal{U}, \rho_2)$ is below the robust frontier of $\mathcal{V}(\mathcal{U}, \rho_1)$.

The main property, used in the above argument, is the fact that the variance of the execution cost does not depend on the impact matrices and uncertainty sets. This property does not hold in the traditional portfolio optimization with an uncertain covariance matrix whose uncertainty set is non-separable from the mean uncertainty set. In such cases, equalities (63) and (64) fail.

8 Concluding remarks

Optimal portfolio execution is currently an important problem for the financial institutions. Amongst its many modeling and computational challenges, estimating temporary and permanent impact matrices (or more generally price impact functions) remains to be one of the most difficult tasks.

To address estimation risk in impact matrices, we consider the robust optimization for an optimal portfolio execution problem. The minimax robust optimization can provide an optimal worst case performance guarantee. Effectiveness of the robust optimization, however, depends on specification of the uncertainty set, which is often imprecise. An uncertainty set with a large size can yield an overly conservative solution.

In addition, we illustrate that the robust execution strategy can be sensitive to the specification of the uncertainty set. Specifically, sensitivity of the robust execution strategy to the upper bound of an interval uncertainty set for the permanent impact matrix can be more severe than the sensitivity of the nominal execution strategy to the nominal impact matrices.

Motivated by the sensitivity analysis for the optimal execution strategy of a nominal optimal execution problem in [42], we propose a regularized robust optimization framework for the considered optimal portfolio execution problem. By imposing a regularization constraint to bound the minimum eigenvalue of the Hessian of the objective function in the problem, we show both mathematically and computationally that sensitivity of the regularized robust execution strategy is significantly improved.

We propose an efficient method based on convex optimization for the regularized robust execution problem. While a robust execution strategy cannot be easily computed for some uncertainty set (e.g., interval uncertainty set), the regularized robust execution strategy can be efficiently derived using convex programming. Finally we analyze mathematically and computationally several implications of regularization on the execution strategy and its corresponding cost.

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