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EXECUTION COSTS IN FINANCIAL MARKETS WITH SEVERAL INSTITUTIONAL INVESTORS

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ABSTRACT
We study multi-period trading strategies of institutional investors who plan to trade the same security during some *finite* time horizons. Investors who trade large volumes face a price impact that depends on their trading volumes simultaneously, and is usually represented as a function, the so called *price-impact function*. We show through a numerical example that a trading strategy, optimal for trading in isolation, may become suboptimal in the presence of other institutional investors who trade the same security at the same time. Thus, the trading activities of other investors should not be ignored in practice and need to be modeled properly. Under the assumptions that the number of other investors and their trading volumes are known, the problem can be modeled as a simultaneous game. We investigate the properties of the equilibrium trading strategies and prove that, under mild assumptions on the price-impact function, an equilibrium uniquely exists and can be computed efficiently. Particularly, when short selling is allowed, the equilibrium is found by solving a system of linear equations. Finally, we evaluate the expected execution cost of the equilibrium trading strategy through simulations and demonstrate that even when other investors choose their trading strategies at random, the expected execution cost of the equilibrium trading strategy is likely to be less than the expected execution cost of the trading strategy that was optimal in the absence of other investors.

KEY WORDS
Algorithmic Trading, Execution Cost, Trading Strategy, Multi-investor Markets, Institutional Investors.

1 Introduction

Large trading volumes of institutional investors exert a non-negligible impact on the execution costs of their trades. A considerable proportion of this impact arises from liquidity costs and information effects derived from large sizes of the trades. Liquidity costs include the cost paid by a purchase initiator (sell initiator) to identify potential sellers (buyers) and result in an instantaneous impact on the execution price that is called the *temporary impact*. Furthermore, the imbalance between supply and demand usually transmits some information to the market that may cause a *permanent impact* on the future execution prices. The sum of the temporary and permanent impacts determines the total price impact incurred by an institutional investor.

Distinctions between temporary and permanent price impacts and their characteristics have been addressed broadly in the literature [10]. A common result is that the magnitude of the price impact is a function of the trading volume, the so called *price-impact function*.

Institutional investors do recognize the price impact of their trades and its dependence on the trading volume. In order to reduce these price impacts, typically institutional investors split their trades into several smaller partial orders, *packages*, and submit the orders during some *fixed* finite number of periods. A sequence of orders submitted during the periods is called a *trading strategy*. There are many possible trading strategies to execute a trade. For a given price-impact function, the *execution cost problem* deals with finding a trading strategy that minimizes the expected execution cost of the trade.

When only one block of equity is traded in the market, the execution cost that should be paid by the investor only depends on his own trade. Consequently, the execution cost problem is reduced to a single-agent decision making problem, i.e., an optimization problem. However, institutional trading does not occur in isolation and all of the institutional trades collectively influence the current and future execution prices. A trading strategy taken by an investor not only induces his own execution cost but also affects other investors' execution costs. In Section (3) we show that other institutional trades may cause a trading strategy, that used to be optimal in the absence of other investors, to become suboptimal.

While each institutional investor's execution cost depends on other investors' trading processes, he is not well-informed about them. Hence, from each institutional investor's point of view other investors' trading strategies are uncertain and the execution cost problem turns into a multi-agent decision making problem. This motivated many researchers to investigate the institutional trading management in a game-theoretic setting and use the language of game theory to discuss the problem. Most of these works model the problem as a *dynamic game* in which decisions are made gradually over an *infinite* number of periods. Moreover, these models usually make some assumptions about trading partners.

In this paper, we model the execution cost problem in a multi-investor market as a *simultaneous game* without making any assumption about the investors' trading partners. In contrast to the models using dynamic games, in our model decisions on the sizes of the packages are made simultaneously before starting to trade. This is of interest particularly when trading time horizons are short. We analyze the equilibrium trading strategies of the generated game and show that for many price-impact functions, the equilibrium is unique and can be computed efficiently. We prove that the equilibrium of the generated game is robust with respect to other institutional investors' actions for strategic investors. Finally, we evaluate the potential performance of the equilibrium trading strategy by a set of simulations. Not only is the equilibrium trading strategy the best response to other strategic investors, it also performs well against *random investors* who place orders of random amounts at each period. In this work, we focus solely on the single-security case and leave the investigation of trading portfolios in multi-investor markets for future work. Throughout this paper, by an investor we mean an institutional investor.

The paper is organized as follows. Section (2) introduces the execution cost problem in single-investor markets as an optimization problem. Section (3) examines how the results for the single-agent models may change when some other investors are trading the same security in the market. A game-theoretic model of the problem in a fixed finite time horizon is presented in Section (4). Some properties of the equilibrium trading strategies are investigated in Section (5) and their performance are evaluated in Section (6). This paper is concluded in Section (7).

2 Execution Costs in Single-Investor Markets

Consider a financial market in which an investor plans to trade $\bar{S}$ shares of some security during a given finite time horizon T. The investor begins his trade at time 0 and his program must be completed by time T. Throughout, without loss of generality, we assume that time is measured in discrete intervals of unit length. Therefore, we may consider that the security is traded over T periods. Although there is an asymmetry in the overall impact of buys and sells (for instance see [11]), their mathematical models are similar. Here, we assume that the investor's goal is to purchase the block of equity. Denote the number of shares traded at the period t by S_t. Positive (negative) S_t implies that the security has been bought (sold) in period t. A trading strategy $S = (S_1, ..., S_T)$ is feasible if $\sum_{t=1}^{T} S_t = \bar{S}$ which guarantees that the trade is finished at the end of the time horizon. Let P_t be the execution price of one share of the security at period t, and this usually follows a stochastic process. The deterministic initial security price per share before the trade begins executing is denoted by P_0, computed from the latest quote just preceding the first price impact. The execution price at each period is determined through some price-impact function. In this paper, we build on price-impact functions that are linear in the trading volume. Linear price-impact functions are well-studied in the literature, for instance see [3, 5, 6, 10]. Furthermore, Huberman et al. [10] demonstrate that only linear permanent price-impact functions rule out arbitrage. These reasons motivated us to use linear price-impact functions $f(S) = \sum_{i=1}^{t} \beta_i^t S_i$ for our investigation. Therefore, the execution price dynamic model is

$$P_t(S) = P_0 + \sum_{i=1}^{t} \beta_i^t S_i + \sigma \sum_{i=1}^{t} \xi_i \text{ for } t = 1, ..., T, \quad (2.1)$$

where σ represents the volatility of the security and ξ_i's are independent zero-mean Gaussian random variables. For every $t = 1, ..., T$ and $i \leq t-1$, nonnegative β_i^t quantifies the permanent price impact and the coefficient $\beta_t^t \geq 0$ measures the importance of the temporary impact. Larger magnitude of the temporary impact relative to the size of the permanent impact — which has been observed in empirical studies— implies that for every $t = 1, ..., T$ and $i \leq t-1$, $\beta_t^t \geq \beta_i^t$. Therefore in single-investor markets, an investor may find an execution cost efficient trading strategy by minimizing the expected value of the execution cost. Then the investor's optimization problem is

$$\min_{S} \mathbb{E}\left(\sum_{t=1}^{T} P_t(S) S_t\right) \text{ s.t.} \sum_{t=1}^{T} S_t = \bar{S}, \;\; S_t \geq 0 \;\; t = 1, ..., T, \quad (2.2)$$

where the nonnegativity constraints appear only when short selling is not allowed. Some other technical constraints may also be added to the problem, and some measure of risk, e.g., variance of costs, can be incorporated in the objective function. For general linear price-impact functions, the optimization problem (2.2) is a quadratic programming problem that can be solved by available optimization software. When short selling is allowed and for every $t = 1, ..., T$, $\beta_t^t = \beta$ and for every $i < t$, $\beta_i^t = \alpha$, for some constants $\alpha \leq \beta$, the unique optimal solution of Problem (2.2) is the *Naive strategy*, that is dividing the total order into identical packages, i.e., $S_t = \frac{\bar{S}}{T}$ for $t = 1, ..., T$. For a detailed discussion about the existence and uniqueness of the optimal trading strategy for the execution price dynamic model (2.1) see [9]. Throughout this paper, we call an optimal solution of Problem (2.2) *an optimal trading strategy*. We use this terminology in the context of multi-investor markets to refer to a trading strategy that has been obtained by ignoring the price impact of other investors.

Most of the existing literature on the execution cost problem focus on markets where only one investor trades (for instance see [3, 4, 5, 6, 8, 9]). These works analyze the market dynamics through the optimization problem (2.2) without solving for an equilibrium.

To find an expected execution cost efficient trading strategy, an investor may simply ignore the effect of other investors and solve a single-agent decision making problem (2.2). However as we show in Section (3) an optimal trading strategy, obtained through Problem (2.2), may not remain optimal when other investors enter the market and start to trade in the same security. It is worthy of notice when the investor knows the probability distributions of other investors' trading strategies, he can incorporate that

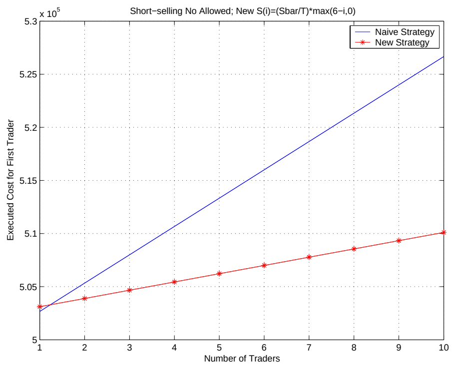

FIGURE 3.1: *Expected execution cost of the Naive strategy versus the strategy* $S_i = \left(\frac{10000}{15}\max(6-i,0)\right)$ *for* $i = 1, 2, ..., 15$*, when short selling is not allowed.*

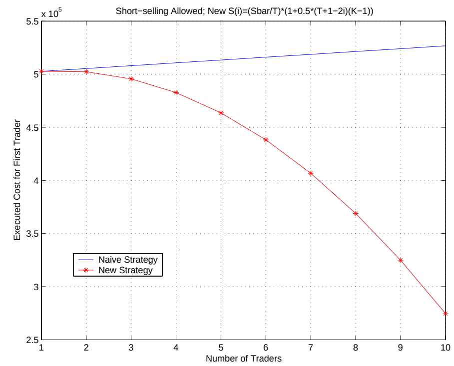

FIGURE 3.2: *Expected execution cost of the Naive strategy versus the strategy* $S_i = \frac{10000}{15}\left(1 + \frac{16-2i}{2(K-1)}\right)$ *for* $i = 1, 2, ..., 15$*, when short selling is allowed.*

information into the problem (2.2). In this case, a cost efficient trading strategy can still be found by means of an optimization problem similar to Problem (2.2), although other active investors do exist in the market. However, it is unlikely that an investor has such exact information about the behavior of other investors.

3 Execution Costs in Multi-Investor Markets

In this section, we show through an example that the optimal solution of Problem (2.2) may be suboptimal when other investors trade the same security at the same time. Consider a market in which $K-1$ investors are simultaneously submitting orders to buy $10,000$ shares of some security, whose current execution price is $50\$/\text{share}$, during 15 periods. The magnitude of the permanent and temporary impacts are identical for all of the investors over the periods, i.e., $\beta_i^t = 5 \times 10^{-5}$ $\$/(\text{share})^2$ for $t = 1, ..., 15$ and $i \leq t$. Assume each of these investors follows the optimal solution of Problem (2.2), the Naive strategy.

Now assume another investor arrives at the market to initiate to buy $10,000$ shares of the same security during 15 periods. Figures (3.1) and (3.2) illustrate when other active investors do exist in the market, i.e., $K \geq 2$, the expected execution cost of another trading strategy is less than the expected execution cost of the Naive strategy.

Thus the trading activities of other investors may considerably reduce the performance of the trading strategy that was optimal in the absence of other investors. The main reason is that in multi-investor markets, the price movement depends simultaneously on all of the investors' actions. These actions should be taken into account when an individual investor seeks a trading strategy with an efficient expected execution cost.

Trading strategies and dynamic behavior of investors in multi-investor settings have been widely considered as a dynamic game in the literature. In these frameworks *strategically interacting investors* choose trading strategies that affect current and future prices of the securities and various equilibrium concepts have been investigated. For a literature review on the available equilibrium approaches, see [13]. Most of these equilibria hold under the assumption that the time horizon is infinite. Moreover these approaches usually set some restrictions on the trading partners. In Section (4), we formulate the execution cost problem as a non-cooperative *simultaneous* game in which investors trade during a *finite* number of periods and choose a trading strategy before starting to trade. The model allows any number of investors. To the best of our knowledge, this is the first static model of the execution cost problem during a *finite* number of periods in a multi-investor market.

4 Robust Equilibrium for the Execution Cost Problem

Let there be K investors going to submit market orders to trade $\bar{S}^{(1)}, \bar{S}^{(2)}, ..., \bar{S}^{(K)}$ shares of the same equity simultaneously. For $k = 1, ..., K$, suppose the trading program of the kth investor must be completed by time T_k (consequently during T_k periods). Denote the kth investor's trading strategy by $S^{(k)}$ that is the T_k-tuple $(S_1^{(k)}, ..., S_{T_k}^{(k)})$. For the sake of simplicity in expression, we denote the trading strategies chosen by all of the investors except the kth investor by $S^{(-k)}$. Similar to (2.1) for the linear price-impact function $f(S^{(1)}, ..., S^{(K)}) = \sum_{m=1}^{K}\sum_{i=1}^{t}\beta_i^t(m)S_i^{(m)}$ the execution price dynamic model in the multi-investor setting can be stated as

$$P_t(S^{(1)}, ..., S^{(K)}) = P_0 + \sum_{m=1}^{K}\sum_{i=1}^{t}\beta_i^t(m)S_i^{(m)} + \sigma\sum_{i=1}^{t}\xi_i, \quad (4.1)$$

where ξ_i's are zero-mean random variables and σ represents the volatility of the underlying security. The coefficient $\beta_i^t(k)$ quantifies the effect of the kth investor's trade, executed in period i, on the execution price at period t. If the kth investor's goal is to buy (sell) the security, then $\beta_i^t(k) \geq 0$ ($\beta_i^t(k) \leq 0$). Hence, the expected execution cost of the kth investor's trade, $\mathbb{E}\left(\sum_{t=1}^{T_k} P_t(S^{(1)}, ..., S^{(K)})S_t^{(k)}\right)$, equals

$\sum_{t=1}^{T_k} S_t^{(k)} \left(P_0 + \sum_{m=1}^{K} \sum_{i=1}^{t} \beta_i^t(m) S_i^{(m)}\right)$, that is denoted by $\varphi_k(S^{(1)}, ..., S^{(k)}, ..., S^{(K)})$ or for simplicity $\varphi_k(S^{(k)}, S^{(-k)})$. Therefore the problem of buying a block of equity during T periods in the presence of $K-1$ other investors can be viewed as a non-cooperative simultaneous game whose players are investors and their trading strategies are counted as players' strategies. Thus the action space of the kth player (investor) is the set of all feasible trading strategies available to him:

$$\mathcal{M}^k := \left\{ (S_1^{(k)}, ..., S_{T_k}^{(k)}) : \sum_{t=1}^{T_k} S_t^{(k)} = \bar{S}^{(k)} \right\},$$

that potentially has infinitely many elements. The set $\mathcal{M}^k$ may also include some nonnegativity constraints if short selling is not allowed. In this simultaneous game, investors choose their trading strategies without knowing those selected by other investors. Moreover the game is played only once and a decision on a trading strategy is made before submitting the first partial order.

For modeling purposes in this paper, we assume that the market characteristics including the number of investors K, the sizes of their blocks and the execution price dynamic model (4.1), that depends on all of the investors' trading sizes simultaneously, are known by every investor. This assumption is not restrictive. We make this assumption since our goal here is to model the problem as a simultaneous game. As soon as the problem is modeled as a game, existing results in game theory about partial information games can be applied whenever the market characteristics are not fully or correctly known by some of the investors. For a few of these results, applicable in our setting, see [1, 12]. The equilibrium whose property we investigate in this paper is defined as follows:

Definition 4.1. *The collection of trading strategies* $(\hat{S}^{(1)}, ..., \hat{S}^{(k)}, ..., \hat{S}^{(K)})$ *is a Nash equilibrium of the game if for every investor* $k = 1, 2, ..., K$

$$\varphi_k(\hat{S}^{(k)}, \hat{S}^{(-k)}) \leq \varphi_k(S^{(k)}, \hat{S}^{(-k)}),$$

for any trading strategy $S^{(k)}$ *in* $\mathcal{M}^k$.

Throughout, we refer to the trading strategies at Nash equilibrium as the equilibrium trading strategies. As we show in Section (5) under some conditions on the coefficients of the price-impact function the equilibrium is unique. At equilibrium no strategic investor has incentive to change his own trading strategy unilaterally given that no other investor changes its strategy. Although we are more interested in demonstrating that equilibrium trading strategies outperform the Naive trading strategy when other investors trade according to the Naive trading strategy or even at random (see Section (6)), nice properties of the equilibrium when investors trade strategically should not be ignored. The following proposition shows that under the assumption that investors trade strategically which is known to all of them, the equilibrium trading strategies incur the least expected execution cost among the trading strategies that are robust with respect to other investors' actions. For its proof, see Appendix A. As the proof shows the statement of Proposition (4.1) remains true for every convex price-impact function.

Proposition 4.1. *Let the equilibrium trading strategy be unique and assume investors act strategically. Then the equilibrium trading strategy has the least expected execution cost among the strategies that are robust with respect to other investors' trading strategies, i.e., for every* $k = 1, 2, ..., K$

$$\varphi_k(\hat{S}^{(k)}, \hat{S}^{(-k)}) \leq \min_{S^{(k)} \in \mathcal{M}^k} \max_{S^{(-k)} \in \prod_{i \neq k} \mathcal{M}^i} \varphi_k(S^{(k)}, S^{(-k)}),$$

where $(\hat{S}^{(1)}, ..., \hat{S}^{(K)})$ *is the equilibrium trading strategy.*

To conclude this section, we want to emphasize that the model and the solution concept do not rely on the assumptions that investors trade solely with each other or the rest of the investors cannot move the price. Moreover, in contrast to the dynamic game models, a simultaneous game model is able to offer a trading strategy over all of the periods before submitting the first partial order.

5 Computing the Equilibrium Trading Strategies

One of the concerns about any equilibrium concept is how efficiently it can be computed. In this section, we analyze the existence and uniqueness of the equilibrium for the execution price dynamic model (4.1) and show how it can be computed. Our discussion exploits the convexity of the underlying price-impact function. Throughout, we define $\beta_i^t(m) = 0$ whenever $i > T_m$. First assume that none of the investors is interested in short selling during his trading program. Therefore the set of possible trading strategies for the kth investor, $\mathcal{M}^k$, is convex, bounded and includes its boundary. The following proposition provides a condition under which the expected execution cost function of each investor is convex in his own trading strategy.

Proposition 5.1. *Let the execution price at period* t *come from Equation* (4.1)*. Then for every investor* $k = 1, 2, ..., K$ *and for every fixed value of* $S^{(-k)}$*, the expected execution cost function* $\varphi_k(S^{(k)}, S^{(-k)})$ *is convex in* $S^{(k)}$ *if and only if the* $T_k \times T_k$ *matrix*

$$\Omega_k := \begin{pmatrix} 2\beta_1^1(k) & \beta_1^2(k) & \beta_1^3(k) & ... & \beta_1^{T_k}(k) \\ \beta_1^2(k) & 2\beta_2^2(k) & \beta_2^3(k) & ... & \beta_2^{T_k}(k) \\ \vdots & \vdots & & ... & \vdots \\ \beta_1^{T_k}(k) & \beta_2^{T_k}(k) & \beta_3^{T_k}(k) & ... & 2\beta_{T_k}^{T_k}(k) \end{pmatrix} \quad (5.1)$$

is positive semidefinite, i.e., $v^T \Omega_k v \geq 0$*, for every* T_k*-tuple real vector* v*. Moreover,* $\varphi_k(S^{(k)}, S^{(-k)})$ *is strictly convex in* $S^{(k)}$ *if and only if* Ω_k *is positive definite.*

Proof. The matrix Ω_k is the Hessian of the function $\varphi_k(S^{(k)}, S^{(-k)})$. Since $\varphi_k(S^{(k)}, S^{(-k)})$ is a quadratic function, it is convex with respect to $S^{(k)}$ if and only if the Hessian of the function $\varphi_k(S^{(k)}, S^{(-k)})$ is positive semidefinite. Thus $\varphi_k(S^{(k)}, S^{(-k)})$ is convex if and only if Ω_k is positive semidefinite. □

Note that for many values of price-impact parameters, the matrices Ω_k's are positive definite. For instance, for $t = 1, ..., T_k$, let $\beta_t^t(k) = \beta(k)$ and for $i \leq t-1$, $\beta_i^t(k) = \alpha(k)$, where $\beta(k)$ and $\alpha(k)$ are some constant parameters so that $\beta(k) \geq \alpha(k)$ ($\beta(k) > \alpha(k)$), then the matrix Ω_k is positive semidefinite (positive definite).

Convexity and smoothness of the function $\varphi_k(S^{(k)}, S^{(-k)})$ for $k = 1, ..., K$ along with the fact that every $\mathcal{M}^k$ is convex, bounded and includes its boundary imply that the generated game in the presence of short selling constraints belong in a famous class of games, namely *convex games* [14]. It was proven that every convex game does have at least one Nash equilibrium and under some conditions it is unique. For a detailed argument about these conditions for the execution price dynamic model (4.1) see Appendix B. The following corollary is a direct consequence of the discussion in Appendix B for a special case.

Corollary 5.1. *Let all of the investors have the same time horizon T and affect the execution price with the same magnitude, i.e, $\beta_i^t(k) = \beta_i^t$ for every $t = 1, ..., T$ and $i \leq t$. Then the positive definiteness of Ω_k implies that the equilibrium uniquely exists.*

This unique equilibrium can be found by the projected gradient method for convex mathematical programming problems [14]. We would like to emphasize that finding necessary conditions on the parameters of the price-impact function under which the equilibrium is unique remains a subject of ongoing research.

Now suppose that investors are allowed to short sell. Thus, for each investor k, $\mathcal{M}^k$ is unbounded. We project $\mathcal{M}^k$ onto $\mathbb{R}^{(T_k-1)}$ by setting $S_{T_k}^{(k)} = \bar{S}^{(k)} - \sum_{t=1}^{T_k-1} S_t^{(k)}$. Therefore, $\varphi_k(S^{(k)}, S^{(-k)})$ can be restated in terms of the first $T_k - 1$ order sizes:

$$
\begin{aligned}
&\varphi_k(S^{(1)}, ..., S^{(k)}, ..., S^{(K)}) \\
&= \sum_{t=1}^{T_k-1} S_t^{(k)} \left(P_0 + \sum_{m=1}^{K} \sum_{i=1}^{t} \beta_i^t(m) S_i^{(m)} \right) \\
&+ \left(\bar{S}^{(k)} - \sum_{j=1}^{T_k-1} S_j^{(k)} \right) \left(\sum_{m=1}^{K} \sum_{i=1}^{T_k-1} \beta_i^{T_k}(m) S_i^{(m)} \right) \\
&+ \beta_{T_k}^{T_k}(k) \left(\bar{S}^{(k)} - \sum_{j=1}^{T_k-1} S_j^{(k)} \right)^2 \\
&+ \left(\bar{S}^{(k)} - \sum_{j=1}^{T_k-1} S_j^{(k)} \right) \left(P_0 + \sum_{m=1, m\neq k}^{K} \beta_{T_k}^{T_k}(m) S_{T_k}^{(m)} \right).
\end{aligned}
$$

Applying this expression of φ_k's, we may use the following well known result in game theory, namely *Equilibrium Test* (for instance see [2]):

Proposition 5.2. *Let the collection of trading strategies $\hat{S} = (\hat{S}^{(1)}, ..., \hat{S}^{(K)})$ satisfy the following conditions:*
(1) *For every investor k, the gradient of the expected execution cost function with respect to the kth investor's order size in period $\bar{t}$, at point $\hat{S}$ is zero, i.e., $\frac{\partial \varphi_k}{\partial S_{\bar{t}}^{(k)}}(\hat{S}) = 0$ for $\bar{t} = 1, ..., T_k - 1$.*
(2) *For every k, $\hat{S}^{(k)}$ is the unique trading strategy that minimizes the expected execution cost of the kth investor, when other investors follow the trading strategies $\hat{S}^{(-k)}$.*
(3) *The Hessian matrices of the expected execution cost functions φ_k's at point $\hat{S}$ are positive definite.*
Then $\hat{S}$ is the unique Nash equilibrium of the generated game.

For every investor k, when other investors follow trading strategy $S^{(-k)}$ the problem of minimizing $\varphi_k(S^{(k)}, S^{(-k)})$ with respect to $S^{(k)} \in \mathcal{M}^k$ is a quadratic programming problem. Hence, positive definiteness of Ω_k implies that its optimal minima is unique. Therefore, when for every $k = 1, ..., K$ the matrix Ω_k is positive definite, the condition (2) of Proposition (5.2) is satisfied. Moreover when Ω_k's are positive definite, the condition (3) holds at every $(S^{(1)}, ..., S^{(K)})$. Thus the solution of the system

$$
\frac{\partial \varphi_k(S^{(k)}, S^{(-k)})}{\partial S_t^{(k)}} = 0, \forall k = 1, \ldots, K, \ \forall t = 1, \ldots, T_k, \quad (5.2)
$$

is the unique Nash equilibrium for the generated game when short selling is allowed. Applying the model (4.1) the above system is reduced to a system of linear equations. This discussion is summarized in the following proposition. For convenience, we state the following proposition for the case that investors have identical time horizons.

Proposition 5.3. *For every $k = 1, ..., K$, let $T_k = T$ and short selling be allowed. Moreover assume the execution price at every period $t = 1, ..., T$ comes from* (4.1) *so that the matrices Ω_k's are positive definite. Then the unique equilibrium is derived from the following system:*

$$
\begin{aligned}
&\sum_{i=1}^{\bar{t}-1} \left[\beta_i^{\bar{t}}(k) - \beta_i^T(k) - \beta_{\bar{t}}^T(k) + 2\beta_T^T(k) \right] S_i^{(k)} \qquad (5.3) \\
&+2 \left[\beta_{\bar{t}}^{\bar{t}}(k) - \beta_{\bar{t}}^T(k) + \beta_T^T(k) \right] S_{\bar{t}}^{(k)} \\
&+ \sum_{i=\bar{t}+1}^{T-1} \left[\beta_{\bar{t}}^i(k) - \beta_i^T(k) - \beta_{\bar{t}}^T(k) + 2\beta_T^T(k) \right] S_i^{(k)} \\
&+ \sum_{m=1, m\neq k}^{K} \sum_{i=1}^{\bar{t}} \left[\beta_i^{\bar{t}}(m) + \beta_T^T(m) - \beta_i^T(m) \right] S_i^{(m)} \\
&+ \sum_{m=1, m\neq k}^{K} \sum_{i=\bar{t}+1}^{T-1} \left[\beta_T^T(m) - \beta_i^T(m) \right] S_i^{(m)} \\
&= \bar{S}^{(k)} \left[2\beta_T^T(k) - \beta_{\bar{t}}^T(k) \right] + \sum_{m=1, m\neq k}^{K} \beta_T^T(m) \bar{S}^{(m)}, \\
&\textit{for every } k = 1, 2, ..., K, \ \bar{t} = 1, 2, ..., T-1.
\end{aligned}
$$

Proof. A set of trading strategies $(S^{(1)}, ..., S^{(K)})$ satisfies Equation (5.3) if and only if it satisfies (5.2). □

Thus when investors move the price with the same magnitude, i.e., $\beta_i^t(k) = \beta_i^t$ for every $k = 1, ..., K$, all an investor needs to know in order to compute the Nash equilibrium is the *total* trading volume of other investors.

We use the result of Proposition (5.3) in our simulation.

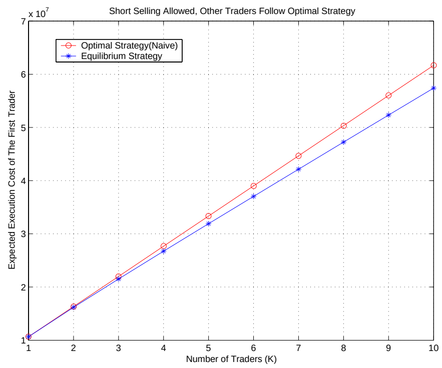

FIGURE 6.1: *Expected execution cost of the equilibrium strategy versus the Naive strategy when $(k-1)$ other institutional investors follow the Naive strategy.*

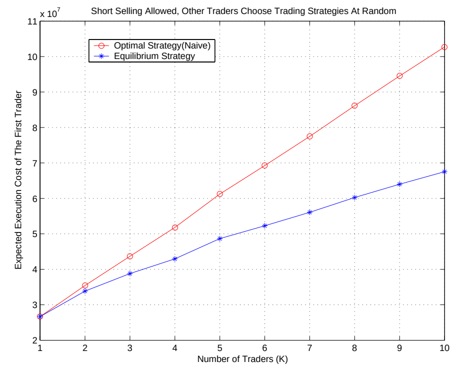

FIGURE 6.2: *Expected execution cost of the equilibrium strategy versus the Naive strategy when $(k-1)$ other institutional investors trade at random for* 1000 *simulations.*

6 Evaluation of the Equilibrium Trading Strategies

In this section, we compare the expected execution cost of the equilibrium trading strategy with the optimal solution of Problem (2.2), the Naive strategy, through simulated investors. We restrict our simulation to the case that short selling is allowed. The setup of the simulation is as follows. We assume $K = 10$ investors commence buying a block of some security whose current execution price is $P_0 = 50\$/\text{share}$. For $k = 1, ..., K$, we set $\bar{S}^{(k)} = 100,000$ shares, $T_k = 15$, $\sigma = 0.95$ $(\$/\text{share})/\text{unit of time}$, and for $t = 1, 2, ..., T_k$, $i \leq t-1$ we define the temporary impact $\beta_t^t(k) = 5 \times 10^{-3}\ \$/\text{share}^2$ and the permanent impact $\beta_i^t(k) = 5 \times 10^{-4}\ \$/\text{share}^2$. Therefore, the execution price follows the following stochastic process:

$$P_t = 50 + 5(10^{-4}) \sum_{k=1}^{K} \sum_{j=1}^{t-1} S_j^{(k)} + 5(10^{-3}) \sum_{k=1}^{K} S_t^{(k)} + \sigma \sum_{j=1}^{t} \xi_j.$$

When each investor ignores the trading activities of other investors, he follows the optimal solution of Problem (2.2), with the execution price dynamic model (2.1), and hence places the partial orders according to the Naive strategy.

In the first example, we assume the investors $k = 2, ..., K$ follow the Naive strategy. As Figure (6.1) depicts under this assumption about other investors, the first investor incurs less expected execution cost by following the equilibrium trading strategy than that of the Naive strategy. Studying the economic behavior of agents under the assumption that the agents' actions are randomly distributed is a fairly standard approach in many dynamic models [7] of the real world markets. This motivated us to carry out our second set of experiments under the assumption that other investors choose their trading strategies at random. The setup of the market is the same as before. We ran M simulations. At a single simulation, we generated $K-1$ random permutations $R^{(k)}$ of the number of periods $2, ..., T$. Then for $k = 2, ..., K$, we defined the random trading strategy $S_i^{(k)} = \ell * \mathsf{rand} * \mathsf{randn} * \bar{S}^{(k)}$, where $\ell = 4$ for $1 \leq R^{(k)}(i) \leq 5$, $\ell = 3$ for $6 \leq R^{(k)}(i) \leq 10$ and $\ell = 5$ for $6 \leq R^{(k)}(i) \leq 14$, with $S_T^{(k)} = \bar{S}^{(k)} - \sum_{j=1}^{T-1} S_j^{(k)}$.

For each set of $K-1$ generated trading strategies, we computed the expected execution cost of the first investor, when he follows the Naive strategy, and his expected execution cost of the equilibrium trading strategy. Then we computed the average of the obtained expected execution costs over M simulations. Figure (6.2) illustrates the average expected execution costs for $M = 1000$.

We ran another set of simulations when investors trade at random with Gaussian distribution, i.e., for $k = 2, ..., K$, we set $S_i^{(k)} = \mathsf{randn} * \bar{S}^{(k)}$ for $i = 1, ..., T-1$ and $S_T^{(k)} = \bar{S}^{(k)} - \sum_{j=1}^{T-1} S_j^{(k)}$. Figure (7.1) depicts the first investor's average expected execution costs over 1000 single simulations by following the Naive strategy versus the random trading strategy.

As the graphs demonstrate, the average of the first investor's expected execution costs over M realizations of the market is less when he follows the equilibrium trading strategy compared to that of the Naive strategy. Therefore, as far as the expected execution cost is concerned, it is *likely* that the performance of the equilibrium trading strategy dominates the performance of the Naive strategy.

7 Conclusion and Some Directions for Future Work

We studied the execution cost problem for risk neutral investors in multi-investor markets with linear price-impact functions. We showed through an example that each investor must take into account other investors' total trading volume when deciding on a trading strategy, otherwise a suboptimal trading strategy may be chosen. We apply game theory to propose a model for this situation. The Nash equilibrium of the generated game is numerically computable. Our simulations illustrate that the expected execution cost of the equilibrium trading strategy is likely to be less than the expected execution cost of the optimal trading strategy.

There remains a need to strengthen the theoretical aspects of the properties of the Nash equilibrium for more general price-impact functions. Moreover exploring the performance of other solution concepts in game theory may

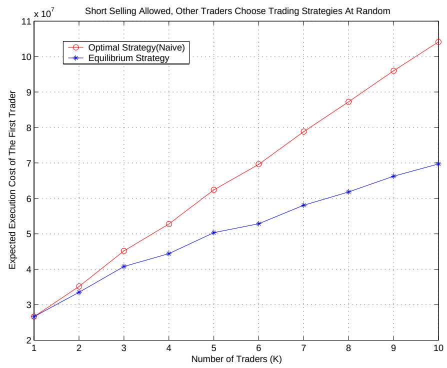

FIGURE 7.1: *Expected execution cost of the equilibrium strategy versus the Naive strategy when* $(k-1)$ *other institutional investors trade at random with Gaussian distribution for* 1000 *simulations.*

result in more interesting contributions, particularly those solution concepts that rely on weaker assumptions about the investors. We also hope to extend our work to portfolio trading.

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Appendix A

Here we prove Proposition (4.1). Let $(\hat{S}^{(1)}, ..., \hat{S}^{(K)})$ be the unique equilibrium trading strategy. Thus for every $S^{(k)} \in \mathcal{M}^k$, $\max_{\hat{S}^{(-k)} \in \prod_{i=1, i\neq k}^{K} \mathcal{M}^i} \varphi_k(S^{(k)}, S^{(-k)})$ is no less than $\varphi_k(S^{(k)}, \hat{S}^{(-k)})$. Therefore, $\min_{S^{(k)} \in \mathcal{M}^k} \max_{\hat{S}^{(-k)} \in \prod_{i=1, i\neq k}^{K} \mathcal{M}^i} \varphi_k(S^{(k)}, S^{(-k)})$ is greater than or equal to $\min_{S^{(k)} \in \mathcal{M}^k} \varphi_k(S^{(k)}, \hat{S}^{(-k)})$. This inequality along with the fact that $(\hat{S}^{(1)}, ..., \hat{S}^{(K)})$ is the unique equilibrium, i.e., $\min_{S^{(k)} \in \mathcal{M}^k} \varphi_k(S^{(k)}, \hat{S}^{(-k)}) = \varphi_k(\hat{S}^{(k)}, \hat{S}^{(-k)})$ proves the statement of Proposition (4.1).

Appendix B

Let $S = (S^{(1)}, ..., S^{(k)}, ..., S^{(K)})$ be a collection of trading strategies for the investors. For each fixed K-tuple real vector $(r_1, ..., r_K)$, define the function $g(S, r_1, ..., r_K)$ in terms of the gradients $\nabla_k \varphi_k(S^{(k)}, S^{(-k)})$ as follows:

$$g(S, r_1, ..., r_K) = \begin{pmatrix} r_1 \nabla_1 \varphi_1(S^{(1)}, S^{(-1)}) \\ r_2 \nabla_2 \varphi_2(S^{(2)}, S^{(-2)}) \\ \vdots \\ r_K \nabla_K \varphi_K(S^{(K)}, S^{(-K)}) \end{pmatrix}.$$

Denote the Jacobian of $g(S, r_1, ..., r_K)$ with respect to S with $G(S, r_1, ..., r_K)$. Rosen [14] proves that if the matrix $[G(S, r_1, ..., r_K) + G^T(S, r_1, ..., r_K)]$ is positive definite for every S, then the equilibrium satisfying (4.1) is unique. According to the notations in Equation (4.1), the $\sum_{m=1}^{K} T_m \times \sum_{m=1}^{K} T_m$ matrix $G(S, r_1, ..., r_K)$ equals

$$\begin{pmatrix} r_1\Omega_1 & r_1\Lambda_{1,2} & r_1\Lambda_{1,3} & \dots & r_1\Lambda_{1,K} \\ r_2\Lambda_{2,1} & r_2\Omega_2 & r_2\Lambda_{2,3} & \dots & r_2\Lambda_{2,K} \\ \vdots & \vdots & & \dots & \vdots \\ r_K\Lambda_{K,1} & r_K\Lambda_{K,2} & r_K\Lambda_{K,3} & \dots & r_K\Omega_K \end{pmatrix}, \quad (9.1)$$

where the $T_i \times T_j$ matrix $\Lambda_{i,j}$ is defined as follows:

$$\Lambda_{i,j} = \begin{pmatrix} \beta_1^1(j) & 0 & 0 & \dots & 0 \\ \beta_1^2(j) & \beta_2^2(j) & 0 & \dots & 0 \\ \vdots & \vdots & & \dots & \vdots \\ \beta_1^{T_i}(j) & \beta_2^{T_i}(j) & \beta_3^{T_i}(j) & \dots & \beta_{T_j}^{T_i}(j) \end{pmatrix}.$$

When the assumptions in Corollary (5.1) hold, the matrices $\Lambda_{i,j}$ are identical, i.e., $\Omega_k = \Omega$ for some fixed matrix Ω. Therefore, setting $r_1 = r_2 = ... = r_K = 1$, the matrix $[G(S, r_1, ..., r_K) + G^T(S, r_1, ..., r_K)]$ is positive definite if and only if Ω is positive definite.