Stability of Sobolev-Regularized Polynomial Differentiation Matrices

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Abstract. *Differentiation matrices associated with orthogonal polynomial bases play an important role in spectral methods and polynomial-based numerical algorithms, but their norms grow rapidly with the polynomial degree, affecting numerical stability. In this work, we investigate differentiation matrices constructed from Legendre-Sobolev and Chebyshev-Sobolev polynomial bases, where derivative information is incorporated through a Sobolev inner product. We first establish the growth of the classical Legendre and Chebyshev differentiation matrices. We then derive the corresponding Sobolev differentiation matrices in terms of the classical differentiation matrices and associated transformation matrices. Using these factorizations, we obtain norm bounds for the transformation and normalization matrices, and derive an upper bound for the Frobenius norm of the Legendre-Sobolev differentiation matrix, highlighting its dependence on the Sobolev parameter μ. These results characterize how the structure and norm of the differentiation matrix are modified when passing from classical orthogonal polynomial bases to Sobolev orthogonal polynomial bases.*

1 Introduction

Orthogonal polynomial bases, such as Legendre and Chebyshev polynomials, are widely used to represent and approximate functions. For polynomials of degree at most n, differentiation can be represented by an n × n matrix that depends on the chosen basis. Although these matrices have finite norms, their norms increase as the polynomial degree n grows.

Differentiation matrices defined in various polynomial bases arise naturally in applications ranging from numerical solutions of differential equations to feature extraction in signal processing and handwriting recognition.

Classical polynomial differentiation matrices exhibit rapid growth with degree, amplifying high-frequency modes and leading to poor numerical conditioning. To address this issue, Sobolev inner products provide a natural framework by including derivative information directly into the notion of orthogonality. It is therefore important to understand how the structure and growth of differentiation matrices are modified when passing from classical orthogonal polynomial bases to their Sobolev counterparts.

Despite the known benefits of Sobolev formulations, a precise theoretical understanding of how Sobolev regularization affects the growth and stability of differentiation matrices remains incomplete. In particular, there is a lack of explicit characterization of how high-frequency modes are controlled and how the matrix norms scale with polynomial degree and regularization strength. We study how Sobolev regularization modifies the growth of polynomial differentiation matrices. Our main contributions are:

- We derive the growth of the classical Legendre and Chebyshev differentiation matrices.
- We derive explicit factorizations for the Legendre-Sobolev and Chebyshev-Sobolev differentiation matrix.
- We establish norm bounds for the Legendre-Sobolev transformation matrices.
- We derive upper bound for the Frobenius norm of Legendre-Sobolev differentiation matrix.

This work is motivated by our research on polynomial representations for handwriting recognition, where derivative information can provide additional information about local shape and variation. The results presented in this paper help characterize how differentiation matrices and their norms are modified when classical polynomial bases are replaced by their Sobolev counterparts. More generally, this analysis contributes to understanding derivative-aware polynomial representations in computational mathematics and intelligent mathematical software.

1.1 Orthogonal Polynomial Bases

Let $\mathbb{P}_N$ denote the space of real polynomials of degree at most N on the interval $[-1, 1]$. We consider orthogonal polynomial bases with respect to an inner product of the form

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\, w(x)\, dx, \tag{1}$$

where $w(x)$ is a non-negative weight function.

This leads to Legendre polynomials [9], when $w(x) = 1$, and Chebyshev polynomials of the first kind [7], when $w(x) = (1 - x^2)^{-1/2}$.

We consider the following Sobolev inner product [2] for $\mu > 0$.

$$\langle f, g\rangle_\mu = \int_{-1}^{1} f(x)g(x)w(x)\, dx + \mu \int_{-1}^{1} f'(x)g'(x)w(x)\, dx. \tag{2}$$

This leads to Legendre-Sobolev polynomials [6], when $w(x) = 1$, and Chebyshev-Sobolev polynomials [8], when $w(x) = (1-x^2)^{-1/2}$. Compared with the classical inner product, the Sobolev inner product measures similarity not only through the values of the polynomials but also through their derivatives, with the parameter (μ) controlling the contribution of the derivative term.

1.2 Differentiation Matrix

A differentiation matrix provides a finite-dimensional representation of differentiation. Once a polynomial is expressed in a chosen basis, differentiating the polynomial can be performed by applying a matrix to its coefficient vector.

Definition 1 (Differentiation Matrix[3]) *Let $\{\phi_k\}_{k=0}^N$ be an orthonormal basis of $\mathbb{P}_N$. The differentiation matrix $D_N \in \mathbb{R}^{(N+1)\times(N+1)}$ is defined as the matrix representation of the derivative operator $D : f \mapsto f'$ in this basis, with entries $(D_N)_{mk} = \langle \phi_m, \phi_k' \rangle$.*

Equivalently, for any $f \in \mathbb{P}_N$ with expansion $f(x) = \sum_{k=0}^N a_k \phi_k(x)$, its derivative can be expressed as $f'(x) = \sum_{m=0}^N \left(\sum_{k=0}^N (D_N)_{mk} a_k \right) \phi_m(x)$.

Example 2 (Position nearness versus derivative nearness) *Consider the functions $f(x) = 0$, and $g(x) = \varepsilon \sin(kx)$. They are uniformly close in position since $|f(x) - g(x)| \leq \varepsilon$. Thus, a metric based only on function values regards f and g as close when ε is small. However, their derivatives can differ as $|f'(x) - g'(x)| \leq \varepsilon k$.*

Hence, even when the positional difference is small, the derivative difference can be much larger when k is large. The derivative therefore reveals variation that is not apparent from position alone.

The same principle can be expressed for polynomial representations using a differentiation matrix. Let $f(x) = \mathbf{a}^T \mathbf{P}(x)$, and $g(x) = \mathbf{b}^T \mathbf{P}(x)$ be two polynomial curves where $\mathbf{a}, \mathbf{b} \in \mathbb{R}^{N+1}$ be coefficient vectors and $\mathbf{P}(x) = \begin{bmatrix} P_0(x) & P_1(x) & \cdots & P_N(x) \end{bmatrix}^T$ be a polynomial basis.

If the coefficient difference is $\mathbf{a} - \mathbf{b}$, then the difference between the coefficients of their derivatives will be $\mathbf{D}(\mathbf{a} - \mathbf{b})$.

Consequently, a derivative aware measure [4] of nearness of the form $\|\mathbf{a}-\mathbf{b}\|^2 + \mu \|\mathbf{D}(\mathbf{a}-\mathbf{b})\|^2$ requires agreement not only in position but also in derivative information.

For applications such as handwriting trajectories, this distinction is useful. Two curves can be close with respect to their coordinate values while differing in important local shape features, such as tangent direction or the location of corners. Incorporating derivative information can distinguish such curves even when a position-based measure indicates that they are close [2]. Thus, a large differentiation matrix norm reflects the sensitivity of differentiation to geometric information that may be useful in comparing polynomial curves.

The matrix p-norm and Frobenius norm of D_N are defined by

$$\|D_N\|_p = \sup_{\substack{f \in \mathbb{P}_N \\ \|f\|_p = 1}} \|f'\|_p, \qquad \|D_N\|_F = \left(\sum_{m=0}^N \sum_{k=0}^N |(D_N)_{mk}|^2 \right)^{1/2}. \tag{3}$$

2 Differentiation Matrix Analysis

Theorem 3 (Legendre Differentiation Growth) *Let $\mathbf{D}_{leg}$ be the differentiation matrix in the orthonormal Legendre basis $\{\phi_k\}_{k=0}^N$. The differentiation matrix $\mathbf{D}_{leg}$ satisfies*

$$\|\mathbf{D}_{leg}\|_F = \Theta(N^2), \qquad \|\mathbf{D}_{leg}\|_2 \leq \Theta(N^2).$$

Proof. Let $\phi_n(x) = \sqrt{\frac{2n+1}{2}}\, P_n(x)$ be the orthonormal Legendre basis and $\mathbf{D}_{leg} = (d_{n,m})_{n,m=0}^N$ be the differentiation matrix, such that $\phi_n'(x) = \sum_{m=0}^N d_{n,m} \phi_m(x)$.

Now, using the Legendre derivative expansion, $P_n'(x) = \sum_{\substack{0\le m\le n-1 \\ n-m \text{ odd}}} (2m+1)P_m(x)$, we obtain

$$\phi_n'(x) = \sum_{\substack{0\le m\le n-1 \\ n-m \text{ odd}}} \sqrt{(2n+1)(2m+1)}\,\phi_m(x).$$

Therefore

$$\mathbf{D}_{leg} = d_{n,m} = \begin{cases} \sqrt{(2n+1)(2m+1)}, & m<n \text{ and } n-m \text{ is odd}, \\ 0, & \text{otherwise}. \end{cases} \tag{4}$$

The infinity norm will be:

$$\|\mathbf{D}_{leg}\|_\infty = \max_{0\le n\le N} \sqrt{2n+1} \sum_{\substack{0\le m<n \\ n-m \text{ odd}}} \sqrt{2m+1}.$$

Using $\sum_{\substack{0\le m<n \\ n-m \text{ odd}}} \sqrt{2m+1} \sim \theta(n^{3/2})$

$$\|\mathbf{D}_{leg}\|_\infty = \Theta(N^2). \tag{5}$$

The 1-norm is the maximum absolute column sum:

$$\|\mathbf{D}_{leg}\|_1 = \max_{0\le m\le N} \sum_{n=0}^{N} |d_{n,m}| = \max_{0\le m\le N} \sqrt{2m+1} \sum_{\substack{m<n\le N \\ n-m \text{ odd}}} \sqrt{2n+1} = \Theta(N^2). \tag{6}$$

Using (6) and (5), in the standard inequality of Matrix 2 Norm,

$$\|\mathbf{D}_{leg}\|_2 \le \sqrt{\|\mathbf{D}_{leg}\|_1 \|\mathbf{D}_{leg}\|_\infty} \le \Theta(N^2). \tag{7}$$

Now, By definition of the Frobenius norm,

$$\|\mathbf{D}_{leg}\|_F^2 = \sum_{n=0}^{N}\sum_{m=0}^{N} |d_{n,m}|^2 = \sum_{n=1}^{N} \sum_{\substack{0\le m<n \\ n-m \text{ odd}}} \left(\sqrt{2n+1}\sqrt{2m+1}\right)^2$$

Finally,

$$\|\mathbf{D}_{leg}\|_F = \frac{1}{2}\sqrt{N(N+1)^2(N+2)} = \Theta(N^2). \tag{8}$$

Theorem 4 (Chebyshev Differentiation Growth) *Let $\mathbf{D}_C$ be the differentiation matrix associated with the orthonormal Chebyshev basis. The differentiation matrix satisfies*

$$\|\mathbf{D}_C\|_F = \Theta(N^2), \qquad \|\mathbf{D}_C\|_2 \le \Theta(N^2).$$

Proof. Let $\phi_n(x)$ be the orthonormal Chebyshev basis of the first kind and $\mathbf{D}_C = (d_{n,m})_{n,m=0}^{N}$ be the differentiation matrix, such that $\phi_n'(x) = \sum_{m=0}^{N} d_{n,m}\phi_m(x)$.

Using the Chebyshev derivative identity $T_n'(x) = nU_{n-1}(x)$, together with standard chebyshev expansion

$$T_n'(x) = \begin{cases} 2n \displaystyle\sum_{\substack{1\le m<n \\ n-m \text{ odd}}} T_m(x), & n \text{ even}, \\ nT_0(x) + 2n \displaystyle\sum_{\substack{1\le m<n \\ n-m \text{ odd}}} T_m(x), & n \text{ odd}, \end{cases}$$

Using the normalization relation both sides, we get

$$\phi_n'(x) = \begin{cases} 2n \displaystyle\sum_{\substack{1\le m<n \\ n-m \text{ odd}}} \phi_m(x), & n \text{ even}, \\ \sqrt{2}\, n\phi_0(x) + 2n \displaystyle\sum_{\substack{1\le m<n \\ n-m \text{ odd}}} \phi_m(x), & n \text{ odd}. \end{cases}$$

Therefore,

$$d_{n,m} = \begin{cases} \sqrt{2}\, n, & m = 0,\ n \text{ odd}, \\ 2n, & 1 \le m < n,\ n-m \text{ odd}, \\ 0, & \text{otherwise}. \end{cases}$$

For example, for $N = 5$, the differentiation matrix associated with the orthonormal Chebyshev basis is

$$\mathbf{D}_{C,5} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 3\sqrt{2} & 0 & 6 & 0 & 0 & 0 \\ 0 & 8 & 0 & 8 & 0 & 0 \\ 5\sqrt{2} & 0 & 10 & 0 & 10 & 0 \end{bmatrix}$$

The infinity norm is

$$\|\mathbf{D}_C\|_\infty = \max_{0\le n\le N} \sum_{m=0}^{N} |d_{n,m}| = \begin{cases} n(n-1) + \sqrt{2}n & n \text{ odd}, \\ n^2, & n \text{ even}. \end{cases} = \Theta(N^2). \tag{9}$$

The 1-norm is

$$\|\mathbf{D}_C\|_1 = \max_{0\le m\le N} \sum_{n=0}^{N} |d_{n,m}| = \begin{cases} \sqrt{2} \sum_{\substack{1\le n\le N \\ n \text{ odd}}} n & m = 0, \\ \sum_{\substack{m<n\le N \\ n-m \text{ odd}}} 2n, & m \ge 1. \end{cases} = \Theta(N^2). \tag{10}$$

Using the standard inequality with estimates from (10) and (9), we get

$$\|\mathbf{D}_C\|_2 \le \sqrt{\|\mathbf{D}_C\|_1 \|\mathbf{D}_C\|_\infty} = O(N^2). \tag{11}$$

Finally, the Frobenius norm is $\|\mathbf{D}_C\|_F = \frac{N(N+1)}{\sqrt{2}} = \Theta(N^2)$.

3 Legendre-Sobolev Polynomial

Let $S_k^{(\mu)}$ denote the Legendre Sobolev polynomials [1], orthogonal with respect to the weighted inner product defined in (2). According to Theorem 1 in [1], for $n \geq 1$ and $\mu \geq 0$, we have

$$S_n^{(\mu)}(x) = a_n(\mu)\, P_n(x) + \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \big(a_n(\mu) - a_{n+2}(\mu)\big)\, P_{n-2k}(x). \tag{12}$$

where $P_{-2}(x) = P_{-1}(x) = 0$, and

$$a_0(\mu) := 1, \qquad a_v(\mu) = \sum_{k=0}^{\lfloor (v-1)/2 \rfloor} \left(\frac{\mu}{4}\right)^k \frac{(v+2k-1)!}{(2k)!\,(v-2k-1)!}, \quad v \geq 1.$$

Theorem 5 (Legendre-Sobolev Differentiation Growth) *Let $\mathbf{D}_{LS}$ be the differentiation matrix associated with the orthonormal Legendre-Sobolev basis. The differentiation matrix satisfies*

$$\mathbf{D}_{LS} = \left(\mathbf{S}_D^{(\mu)}\right)^{-1} \mathbf{N}^{(\mu)} \mathbf{D}_{\text{leg}} \mathbf{M}_2 \mathbf{S}_D^{(\mu)}, \qquad \|\mathbf{D}_{LS}\|_F \leq \mathcal{O}\left(a_N(\mu)^2 N^3 \sqrt{1+\mu N^2}\right)$$

Proof. Let $\mathbf{S}^{(\mu)} = \begin{bmatrix} S_0^{(\mu)}(x) & S_1^{(\mu)}(x) & \cdots & S_N^{(\mu)}(x) \end{bmatrix}^T, \qquad \mathbf{P} = \begin{bmatrix} P_0(x) & P_1(x) & \cdots & P_N(x) \end{bmatrix}^T$. Using transformation matrix from [1] and eq. (4), the first order derivative can be written as

$$\mathbf{S}'^{(\mu)} = \mathbf{N}^{(\mu)} \mathbf{P}' = \mathbf{N}^{(\mu)} \mathbf{D}_{leg} \mathbf{P} \tag{13}$$

Using transformation matrices from [2], we get

$$\mathbf{S}'^{(\mu)} = \mathbf{N}^{(\mu)} \mathbf{D}_{leg} \mathbf{M}_2 \mathbf{S}^{(\mu)} \tag{14}$$

Let $\mathbf{\Phi}^{(\mu)}$ contains orthonormal Legendre-Sobolev polynomial and $\mathbf{\Phi}'^{(\mu)}$ contains its derivatives,

$$\mathbf{\Phi}'^{(\mu)} = \left(\mathbf{S}_D^{(\mu)}\right)^{-1} \mathbf{N}^{(\mu)} \mathbf{D}_{leg} \mathbf{M}_2 \mathbf{S}_D^{(\mu)} \mathbf{\Phi}^{(\mu)} \tag{15}$$

where, $\mathbf{S}_D^{(\mu)} = \operatorname{diag}\Big(\|S_0^{(\mu)}(x)\|, \|S_1^{(\mu)}(x)\|, \ldots, \|S_N^{(\mu)}(x)\|\Big)$. Finally,

$$\mathbf{D}_{LS} = \left(\mathbf{S}_D^{(\mu)}\right)^{-1} \mathbf{N}^{(\mu)} \mathbf{D}_{\text{leg}} \mathbf{M}_2 \mathbf{S}_D^{(\mu)} \tag{16}$$

where,

$$[\mathbf{M}_2]_{i,j} = \begin{cases} 0, & \text{if } i < j, \\ \dfrac{1}{a_{i-1}(\mu)}, & \text{if } i = j, \\ \dfrac{1}{a_{i-2\ell-1}(\mu)} - \dfrac{1}{a_{i-2\ell+1}(\mu)} = \dfrac{1}{a_{j-1}(\mu)} - \dfrac{1}{a_{j+1}(\mu)}, & \begin{array}{l} \text{if } j = i - 2\ell, \\ \ell = 1, \ldots, \left\lfloor \dfrac{i-1}{2} \right\rfloor, \end{array} \\ 0, & \text{otherwise.} \end{cases}$$

$$
[\mathbf{N}^{(\mu)}]_{i,j} = \begin{cases} 0, & \text{if } i < j, \\ a_{i-1}(\mu), & \text{if } i = j, \\ a_{i-2\ell-1}(\mu) - a_{i-2\ell+1}(\mu) = a_{j-1}(\mu) - a_{j+1}(\mu), & \text{if } j = i - 2\ell,\ \ell = 1, \dots, \left\lfloor \frac{i-1}{2} \right\rfloor, \\ 0, & \text{otherwise.} \end{cases}
$$

Applying the sub-multiplicative property of the Frobenius norm [5] to the factorization of $\mathbf{D}_{LS}$, we obtain the estimate

$$
\begin{aligned}
\|\mathbf{D}_{LS}\|_F &\leq \left\| \left(\mathbf{S}_D^{(\mu)}\right)^{-1} \right\|_2 \|\mathbf{N}^{(\mu)}\|_2 \, \|\mathbf{D}_{\text{leg}}\|_F \, \|\mathbf{M}_2\|_2 \, \|\mathbf{S}_D^{(\mu)}\|_2 \\
&= \kappa\left(\mathbf{S}_D^{(\mu)}\right) \|\mathbf{N}^{(\mu)}\|_2 \, \|\mathbf{D}_{\text{leg}}\|_F \, \|\mathbf{M}_2\|_2
\end{aligned} \tag{17}
$$

where, $\kappa(\mathbf{S}_D^{(\mu)})$ is the condition number of matrix $\mathbf{S}_D^{(\mu)}$.

3.1 Norm bounds for transformation matrix $\mathbf{N}^{(\mu)}$

We know that $\mathbf{N}^{(\mu)} \in \mathbb{R}^{(N+1)\times(N+1)}$ is a lower triangular matrix and $\mu \geq 0$. Since every term in the definition of $a_v(\mu)$ is non-negative, the sequence $\{a_v(\mu)\}_{v\geq 0}$ is non-decreasing.

The infinity norm. The absolute sum of rows corresponding to the row i is

$$
R_i = a_{i-1}(\mu) + \sum_{\ell=1}^{\lfloor (i-1)/2 \rfloor} |a_{i-2\ell-1}(\mu) - a_{i-2\ell+1}(\mu)| = a_{i-1}(\mu) + \sum_{\ell=1}^{\lfloor (i-1)/2 \rfloor} (a_{i-2\ell+1}(\mu) - a_{i-2\ell-1}(\mu))
$$

This simplifies to, $R_i = 2a_{i-1}(\mu) - 1$. Because $a_v(\mu)$ is nondecreasing, the largest row sum occurs for $i = N+1$. Consequently, $\|\mathbf{N}^{(\mu)}\|_\infty = 2a_N(\mu) - 1$.

The one norm. The absolute column sum is

$$
C_j = a_{j-1}(\mu) + \left\lfloor \frac{N+1-j}{2} \right\rfloor (a_{j+1}(\mu) - a_{j-1}(\mu)).
$$

The absolute column sums are non-decreasing and we conclude that $\|\mathbf{N}^{(\mu)}\|_1 = a_N(\mu)$. Applying the standard inequality to $\mathbf{N}^{(\mu)}$ gives

$$
\|\mathbf{N}^{(\mu)}\|_2 \leq \sqrt{\|\mathbf{N}^{(\mu)}\|_1 \|\mathbf{N}^{(\mu)}\|_\infty} = \sqrt{a_N(\mu)\,(2a_N(\mu) - 1)}.
$$

Using $\rho(A) \leq \|A\|_2$, we also obtain the lower bound $a_N(\mu) \leq \|\mathbf{N}^{(\mu)}\|_2$. Combining the lower and upper bounds yields

$$
a_N(\mu) \leq \|\mathbf{N}^{(\mu)}\|_2 \leq \sqrt{a_N(\mu)\,(2a_N(\mu) - 1)}. \tag{18}
$$

3.2 Norm bounds for the inverse transformation matrix $\mathbf{M}_2^{(\mu)}$

We know that, the nonzero entries below the diagonal occur only within the same parity class and all nonzero entries of $\mathbf{M}_2^{(\mu)}$ are nonnegative.

The Infinity norm The absolute sum of row $i = 2r$ and $i = 2r + 1$ will be

$$R_{2r} = b_{2r} + \sum_{\ell=1}^{r} (b_{2r-2\ell} - b_{2r-2\ell+2}) = b_0 = 1$$

$$R_{2r+1} = b_{2r+1} + \sum_{\ell=1}^{r} (b_{2r+1-2\ell} - b_{2r+3-2\ell}) = b_1 = 1$$

where $b_j(\mu) := \frac{1}{a_j(\mu)}$, so that $b_0(\mu) = b_1(\mu) = 1$. Therefore,

$$\left\| \mathbf{M}_2^{(\mu)} \right\|_\infty = \max_{0 \le i \le N} R_i = 1. \tag{19}$$

The 1 norm The diagonal entry in column j is $\frac{1}{a_j(\mu)}$ and the remaining nonzero entries $\frac{1}{a_j(\mu)} - \frac{1}{a_{j+2}(\mu)}$ occur at $i = j + 2,\ j + 4, \dots, j + 2 \left\lfloor \frac{N-j}{2} \right\rfloor$.
The absolute column sum will be

$$C_j = \frac{1}{a_j(\mu)} + \left\lfloor \frac{N-j}{2} \right\rfloor \left(\frac{1}{a_j(\mu)} - \frac{1}{a_{j+2}(\mu)} \right)$$

Taking the maximum over all columns

$$\left\| \mathbf{M}_2^{(\mu)} \right\|_1 = \max_{0 \le j \le N} \left\{ \frac{1}{a_j(\mu)} + \left\lfloor \frac{N-j}{2} \right\rfloor \left(\frac{1}{a_j(\mu)} - \frac{1}{a_{j+2}(\mu)} \right) \right\} \le 1 + \left\lfloor \frac{N}{2} \right\rfloor.$$

The matrix norm of $\mathbf{M}_2^{(\mu)}$ satisfies $\left\| \mathbf{M}_2^{(\mu)} \right\|_2 \le \sqrt{\left\| \mathbf{M}_2^{(\mu)} \right\|_1}$,

$$\left\| \mathbf{M}_2^{(\mu)} \right\|_2 \le \left[1 + \left\lfloor \frac{N}{2} \right\rfloor \right]^{1/2} \tag{20}$$

3.3 Norm bounds for the diagonal matrix $\mathbf{S}_D^{(\mu)}$

We know that $\mathbf{S}_D^{(\mu)}$ is a diagonal matrix. For $n \ge 1$, the squared Legendre-Sobolev norm satisfies

$$\|S_n^{(\mu)}\|_{LS}^2 = a_n(\mu) \left[\frac{2a_n(\mu)}{2n+1} + \mu \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (4n - 8k - 2) a_{n-2k}(\mu) \right].$$

Since all terms in the derivative contribution are non-negative,

$$\|S_n^{(\mu)}\|_{LS}^2 \ge \frac{2a_n(\mu)^2}{2n+1} \ge \frac{2}{2N+1} \tag{21}$$

For the upper bound, monotonicity of $a_n(\mu)$ gives $a_{n-2k}(\mu) \leq a_n(\mu)$. Therefore,

$$\|S_n^{(\mu)}\|_{LS}^2 \leq a_n(\mu)^2 \left[\frac{2}{2n+1} + \mu \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (4n - 8k - 2) \right].$$

The finite sum is $\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (4n - 8k - 2) = n(n+1)$. Thus,

$$\|S_n^{(\mu)}\|_{LS}^2 \leq a_n(\mu)^2 \left[\frac{2}{2n+1} + \mu n(n+1) \right].$$

Finally, using $a_n(\mu) \leq a_N(\mu)$, $n \leq N$, and $2/(2n+1) \leq 2$ yields

$$\|S_n^{(\mu)}\|_{LS}^2 \leq a_N(\mu)^2 \left[2 + \mu N(N+1)\right].$$

The matrix norm bounds for diagonal matrix $\mathbf{S}_D^{(\mu)}$ and its inverse are as follows:

$$\sqrt{2} \leq \|\mathbf{S}_D^{(\mu)}\|_2 \leq a_N(\mu)\sqrt{2 + \mu N(N+1)}$$

and

$$\frac{1}{a_N(\mu)\sqrt{2 + \mu N(N+1)}} \leq \left\| \left(\mathbf{S}_D^{(\mu)}\right)^{-1} \right\|_2 \leq \sqrt{\frac{2N+1}{2}}$$

In particular,

$$\kappa\left(\mathbf{S}_D^{(\mu)}\right) \leq a_N(\mu)\sqrt{\frac{(2N+1)\left(2 + \mu N(N+1)\right)}{2}} \tag{22}$$

3.4 Frobenius Norm bounds for Legendre Sobolev Differentiation matrix

Using (8), (18), (20), and (22) in (17), we obtain

$$\begin{aligned} \|\mathbf{D}_{LS}\|_F \leq{} & a_N(\mu)\sqrt{\frac{(2N+1)\left(2 + \mu N(N+1)\right)}{2}}\sqrt{a_N(\mu)\left(2a_N(\mu) - 1\right)} \\ & \times \frac{1}{2}\sqrt{N(N+1)^2(N+2)}\left(1 + \left\lfloor \frac{N}{2} \right\rfloor\right)^{1/2} \end{aligned} \tag{23}$$

Consequently,

$$\|\mathbf{D}_{LS}\|_F \leq \frac{a_N(\mu)^{3/2}\sqrt{2a_N(\mu) - 1}}{2\sqrt{2}} \left[(2N+1)\left(2 + \mu N(N+1)\right) N(N+1)^2(N+2)\left(1 + \left\lfloor \frac{N}{2} \right\rfloor\right) \right]^{1/2}$$

Finally, we obtain $\|\mathbf{D}_{LS}\|_F \leq \mathcal{O}\left(a_N(\mu)^2 N^3 \sqrt{1 + \mu N^2}\right)$

4 Chebyshev-Sobolev Polynomial

Theorem 6 (Chebyshev-Sobolev Differentiation Growth) *Let $\mathbf{D}_{CS}$ be the differentiation matrix associated with the orthonormal Chebyshev-Sobolev basis. Then*

$$\mathbf{D}_{CS} = \left(\mathbf{Q}_D^{(\mu)}\right)^{-1} \mathbf{N}_C^{(\mu)} \mathbf{D}_{\text{cheb}} \mathbf{M}_{2,C}^{(\mu)} \mathbf{Q}_D^{(\mu)}, \qquad \|\mathbf{D}_{CS}\|_F \leq \kappa \left(\mathbf{Q}_D^{(\mu)}\right) \|\mathbf{N}_C^{(\mu)}\|_2 \, \|\mathbf{D}_{\mathrm{C}}\|_F \, \|\mathbf{M}_{2,C}\|_2$$

Here, $\mathbf{Q}_D^{(\mu)} = \operatorname{diag}\left(\|Q_0^{(\mu)}\|, \|Q_1^{(\mu)}\|, \ldots, \|Q_N^{(\mu)}\|\right)$ is the diagonal norm matrix of the Chebyshev-Sobolev basis, $\mathbf{N}_C^{(\mu)}$ and $\mathbf{M}_{2,C}^{(\mu)}$ are the corresponding transformation matrices, and $\mathbf{D}_{\mathrm{C}}$ is the classical Chebyshev differentiation matrix.

Proof. The proof follows the same steps as in the Legendre-Sobolev case. Using the corresponding Chebyshev-Sobolev transformation matrices and the classical Chebyshev differentiation matrix, we obtain the stated factorization. Applying the mixed sub-multiplicative inequality [5], we obtain the required norm bound.

5 Conclusion

In this paper, we studied the growth of differentiation matrices associated with classical orthogonal polynomials and their Sobolev variants. We first established the growth of the classical Legendre and Chebyshev differentiation matrices. We then derived explicit factorizations for the corresponding Sobolev differentiation matrices and obtained bounds for the associated transformation, normalization, and differentiation matrices. These results provide a theoretical characterization of how the differentiation matrix and its norm are modified when the underlying polynomial basis is defined with respect to a Sobolev inner product. In future work, we plan to investigate sharper bounds for Sobolev differentiation matrices and study their applications in polynomial-based methods for handwriting recognition and other problems in computational mathematics.

References

[1] Alvandi, P., Watt, S.M.: Real-time computation of Legendre–Sobolev approximations. In: Proc. SYNASC. pp. 67–74. IEEE (2018)

[2] Alvandi, P., Watt, S.M.: The LegendreSobolev package and its applications in handwriting recognition. In: Maple Conference. pp. 13–29. Springer (2019)

[3] Amiraslani, A., Corless, R.M., Gunasingam, M.: Differentiation matrices for univariate polynomials. Numerical Algorithms **83**(1), 1–31 (2020)

[4] Corless, R.M., Kalhan, D.S., Watt, S.M.: Well-conditioned polynomial representations for mathematical handwriting recognition. In: Proc. SYNASC. pp. 13–20. IEEE (2025)

[5] Higham, N.J.: Functions of matrices: theory and computation. SIAM (2008)

[6] Hu, R., Watt, S.M.: Determining points on handwritten mathematical symbols. In: Proc. ICICM. pp. 168–183. Springer (2013)

[7] Rivlin, T.J.: Chebyshev polynomials. Courier Dover Publications (2020)

[8] Singh Kalhan, D., Watt, S.M.: A first look at Chebyshev–Sobolev series for digital ink. arXiv e-prints pp. arXiv–2408 (2024)

[9] Sullivan, T.: Orthogonal polynomials and applications. In: Introduction to Uncertainty Quantification, pp. 133–164. Springer (2015)