The Inverse of the Complex Gamma Function

D. J. Jeffrey
Ontario Research Centre for Computer Algebra
and Department of Mathematics
University of Western Ontario, Canada
djeffrey@uwo.ca

Stephen M. Watt
Ontario Research Centre for Computer Algebra
and Cheriton School of Computer Science
University of Waterloo, Canada
smwatt@uwaterloo.ca

***Abstract*—We consider the functional inverse of the Gamma function in the complex plane, where it is multi-valued, and define a set of suitable branches by proposing a natural extension from the real case.**

I. Introduction

Professor James Davenport, whose 70th birthday is being celebrated at this conference, has been a prolific investigator of multivalued functions. His work has covered several topics in the area, dating back more than two decades—see, for example, [1], [2], [3], [4]. The present article follows in this tradition.

A recent review of the Gamma function and its properties comments that the inverse function has received little study [5]. Some basic properties were given in [6] and [7], but the structure of the branches in the complex plane has not been considered. This article provides some results in this area.

The Gamma function for complex z can be defined by [9]

$$\Gamma(z+1) = z! = \int_0^\infty t^z \mathrm{e}^{-t}\,\mathrm{d}t \quad \text{for} \quad \Re z > -1 \ , \tag{1}$$

together with Euler's reflection formula

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin \pi z} \ . \tag{2}$$

The Γ notation shifts the argument with respect to factorial by 1, as shown in (1); this shift, often called a *minor but continual nuisance* [5], is usually blamed on Legendre; but in fact Euler did this first [8]. We denote the inverse of Γ by either $w = \check{\Gamma}_k(z)$, or $w = \mathrm{inv}\Gamma_k(z)$, where k will label the branch, which will be defined here.

We remind the reader of some basic properties of Γ. Figure 1 shows a plot of the Γ-function. The derivative of Γ introduces the digamma function, Ψ, through the logarithmic derivative:

$$\frac{d}{dx}\ln\Gamma(x) = \Psi(x) \ ,$$

thus making $\Gamma' = \Psi\Gamma$. The complex conjugate is $\overline{\Gamma(z)} = \Gamma(\overline{z})$.

In [5], Stirling's approximation is used to give an asymptotic approximation to $\check{\Gamma}_0$ using the Lambert W function.

$$\check{\Gamma}(x) \approx \frac{1}{2} + \frac{\ln\left(x/\sqrt{2\pi}\right)}{W\left(e^{-1}\ln\left(x/\sqrt{2\pi}\right)\right)} \ . \tag{3}$$

This approximation is remarkably accurate, even for relatively small arguments. For example, it gives $\check{\Gamma}(24) \approx 4.99$, where

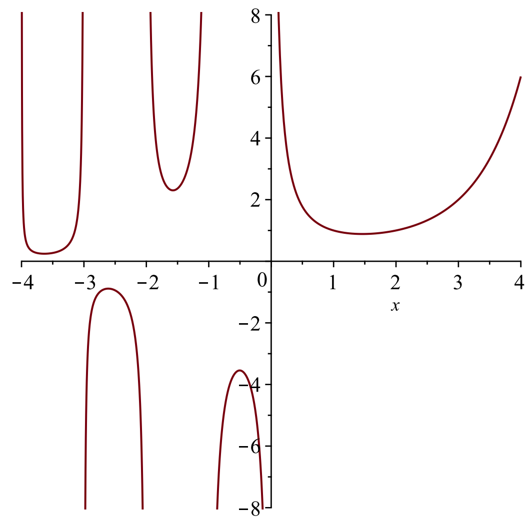

Fig. 1: A graph of the Γ function. Note the poles at non-positive integers.

the exact answer is 5. Trying a smaller value, we see $\check{\Gamma}(4) \approx 3.653$. This can be checked by comparing $\Gamma(3.653) = 3.948$ with 4. Finally, we note that Pedersen [6] has shown that one branch of the inverse can be extended to the complex plane, and it can be represented as a Pick function.

II. Definition of real branches

We first consider the inverse of Γ as a function of the reals. The graph of $\check{\Gamma}$ is show in Figure 2, as produced using the Maple parametric plot command:

```
plot([GAMMA(x),x,x=-4..3.5], discont=true);
```

The turning points of Γ define the boundaries of the branches, and the partition of the range of $\check{\Gamma}$. To define the branches, we introduce some notation. Since the Gamma function has extremal points at

$$\frac{d\Gamma(z)}{dz} = \Gamma(z)\Psi(z) = 0 \ , \tag{4}$$

we denote the points (ψ_i, γ_i) by the definitions

$$\Psi(\psi_0) = 0 \ , \text{ where } \psi_0 > 0 \ , \tag{5}$$

$$\Psi(\psi_k) = 0 \ , \text{ where } k < 0 \text{ and } k < \psi_k < k+1 \ , \tag{6}$$

together with $\gamma_k = \Gamma(\psi_k)$. Numerical values are displayed in Table I. Further values are given in Table 5.4.1 of [9], with our k corresponding to their $-n$.

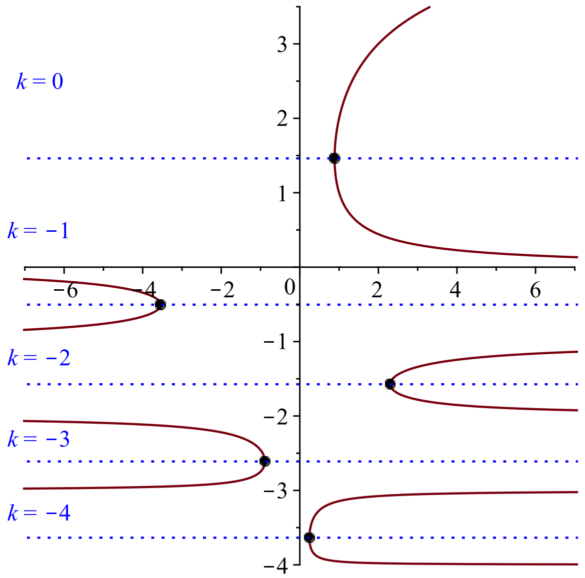

Fig. 2: Inverse of the Γ function showing the critical points and the division of the range into branches.

k	ψ_k	γ_k
0	1.461632	0.885603
-1	-0.504083	-3.544644
-2	-1.573498	2.302407
-3	-2.610720	-0.888136
-4	-3.635293	0.245127
-5	-4.653237	-0.052780

Table I: Critical values for branches of $\check{\Gamma}_k$. These points are plotted as (γ_k, ψ_k) in Figure 2.

k	Range condition	Argument range
0	$\psi_0 \le \check{\Gamma}_0$	$g \ge \gamma_0$
-1	$0 < \check{\Gamma}_{-1} < \psi_0$	$g > \gamma_0$
	$\psi_{-1} \le \check{\Gamma}_{-1} < 0$	$g \le \gamma_{-1}$
-2	$-1 < \check{\Gamma}_{-2} < \psi_{-1}$	$g < \gamma_{-1}$
	$\psi_{-2} \le \check{\Gamma}_{-2} < -1$	$g \ge \gamma_{-2}$

Table II: Branches of inverse real Γ with notation $g = \Gamma(x)$ and $x = \check{\Gamma}(g)$. If g falls outside the intervals shown, there is no real value for inverse Gamma.

The branches are defined by their ranges as displayed in Table II. For the general approach to branched functions, see [10]. The numbering of the branches is chosen so that $k = 0$ is the principal branch, while the remaining branches receive negative numbers so that the labels follow roughly the ranges taken by the branches. Algorithm 1 determines the domain of x values corresponding to a value of $\Gamma(x)$ and branch number.

Some special cases are worth noting. Since $\Gamma(2) = 1! = \Gamma(1) = 0! = 1$, the corresponding inverses must lie on different branches. This requires $\check{\Gamma}_0(1) = 2$ and $\check{\Gamma}_{-1}(1) = 1$. Some other special values of interest are $\check{\Gamma}_0(\sqrt{\pi}/2) = 3/2$, which lies very close to the branch point, and $\check{\Gamma}_{-1}(\sqrt{\pi}) = 1/2$.

III. Extension to the Complex Plane

We follow [11] and [12] in considering extensions to the complex plane. We start with the domain of $\Gamma(z)$, which will become the range of $\check{\Gamma}$. We wish to partition $\mathbb{C}$ so that Γ is injective on each element of the partition. We argue that the parallel lines $z = \psi_k + iy$ partition $\mathbb{C}$ in this way.

Theorem 1. Let $\mathcal{D}_0 = \{z \mid \psi_0 \le \Re z\}$ and $\mathcal{D}_k = \{z \mid \psi_k \le \Re z < \psi_{k+1}\}$ for $k < 0$. Then $\Gamma : \mathcal{D}_k \to \mathbb{C}$ is injective for $k \le 0$.

Proof. We start with the observation that all zeros of Ψ are real, and that Ψ is analytic everywhere except on the non-positive integers. We show we cannot have two points $z_1, z_2 \in \mathcal{D}_k$ with $z_1 \ne z_2$ but $\Gamma(z_1) = \Gamma(z_2)$. Since Γ has no zeros, it is useful to work with the entire function $1/\Gamma$ and we require $1/\Gamma(z_1) = 1/\Gamma(z_2)$. Consider $1/\Gamma$ restricted to $\{z \mid z = L(\lambda) = (1-\lambda)z_1 + \lambda z_2, \lambda \in [0,1]\}$, that is a line in $\mathcal{D}_k$ from z_1 to z_2. Then the real and imaginary parts of $1/\Gamma(L(\cdot))$ are both $C^\infty[0,1]$ real-valued functions so $N(\lambda) = |1/\Gamma(L(\lambda))|^2$ is a differentiable real function on $[0,1]$ with $N(0) = N(1)$. Bearing in mind that $z_1 \ne z_2$, by Rolle's theorem we must

Algorithm 1 Γ Real Domain Selection and Inverse by Index

▷ *Determine the x domain given $g = \Gamma(x)$ and index k.*
▷ *Returns bounds as pair (lo, hi), $lo \in \mathbb{R}$, $hi \in \mathbb{R} \cup +\infty$.*
function RealGammaDomain($g \in \mathbb{R}, k \in \mathbb{Z}_{\le 0}$)
  ▷ *Exclude nonexistent branches.*
  if $k \ge 1$ **or** $g = 0$ **or** ($k = 0$ **and** $g < 0$) **then error**

  ▷ *How much to adjust end points near $k = 0$.*
  if $k = 0$ **then** $\mathrm{lo}_0 \leftarrow 1$; $\mathrm{hi}_0 \leftarrow +\infty$
  else if $k = -1$ **and** $g > 0$ **then** $\mathrm{lo}_0 \leftarrow 0$; $\mathrm{hi}_0 \leftarrow 1$
  else $\mathrm{lo}_0 \leftarrow 0$; $\mathrm{hi}_0 \leftarrow 0$

  ▷ *Determine which side of the pole.*
  if k is even **xor** $g > 0$ **then**
    $\mathrm{lo} \leftarrow k + 1 + \mathrm{lo}_0$
    $\mathrm{hi} \leftarrow$ solve $\Psi(x) = 0$ for $x \in (\mathrm{lo}, k + 2 + \mathrm{hi}_0)$
    if $|g| < |\Gamma(\mathrm{hi})|$ **then error**
  else
    $\mathrm{hi} \leftarrow k + 1 + \mathrm{hi}_0$
    $\mathrm{lo} \leftarrow$ solve $\Psi(x) = 0$ for $x \in (k + \mathrm{lo}_0, \mathrm{hi})$
    if $|g| < |\Gamma(\mathrm{lo})|$ **then error**
  return (lo, hi)

▷ *Compute real inverse Γ on given branch.*
function RealInvGamma($g \in \mathbb{R}, k \in \mathbb{Z}_{\le 0}$)
  $(\mathrm{lo}, \mathrm{hi}) \leftarrow$ RealGammaDomain(g, k)
  return solve $\Gamma(x) = g$ for $x \in (\mathrm{lo}, \mathrm{hi})$

have $N'(\lambda) = 0$ for some $\lambda \in (0,1)$. But $N'(\lambda) = 0$ only when $\Psi(L(\lambda)) = 0$, *i.e.* for $z = \psi_k$, which is the unique point in $\mathcal{D}_k$ for which $\Psi(z) = 0$. So unless the line segment $L : [0,1] \to \mathcal{D}_k$ crosses ψ_k, $\Gamma(z_1) \ne \Gamma(z_2)$.

The segment L will cross ψ_k only if $\Re z_1 = \Re z_2 = \psi_k$. It remains to show that we cannot have $\Gamma(z_1) = \Gamma(z_2)$ in this

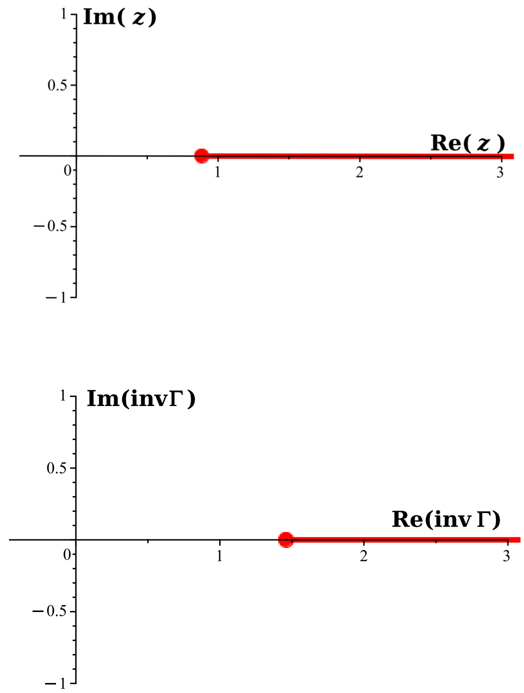

Fig. 3: The domains and ranges for Γ and $\check{\Gamma}$. On the real axes are marked the values taken by the principal branch $\check{\Gamma}_0$.

case. By the same argument as earlier, we cannot have $\Im z_1$ and $\Im z_2$ with the same sign, since that would require a zero of Ψ off the real line. Therefore $\Im z_1$ and $\Im z_2$ must have opposite signs. Without loss of generality, let $z_1 = \psi_k + iy_1$ and $z_2 = \psi_k - iy_2$ with $y_1, y_2 \in \mathbb{R}_{>0}$. We observe that $\Im\Gamma(\phi_k + i\eta)$ must have a fixed sign for all $\eta > 0$ and a fixed sign for all $\eta < 0$. Otherwise, by continuity, Γ would have a real value off the real line, which does not occur. Since

$$\overline{\Gamma(z_2)} = \Gamma(\overline{z_2}) = \Gamma(\psi_k + iy_2),$$

we must have

$$\operatorname{sign} \Im\Gamma(z_1) = \operatorname{sign} \Im\overline{\Gamma(z_2)} = -\operatorname{sign} \Im\Gamma z_2$$

but $\Gamma(z_1) = \Gamma(z_2)$ implies $-\operatorname{sign} \Im\Gamma(z_2) = -\operatorname{sign} \Im\Gamma(z_1)$ so $y_1 = y_2 = 0$ and $z_1 = z_2$, contradicting our hypothesis. $\square$

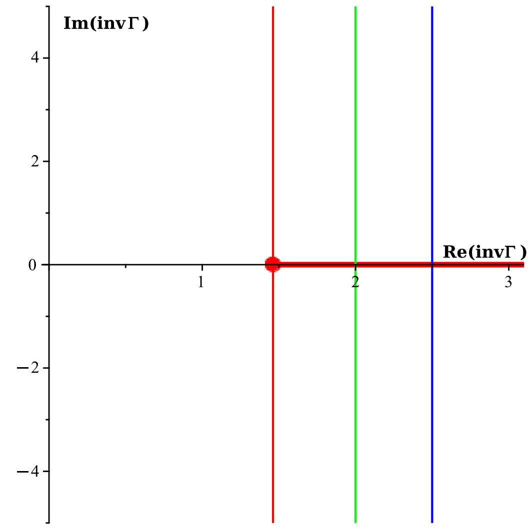

Fig. 4: Contours in $\mathcal{D}_0$ to visualize the range of the $\check{\Gamma}_0$ principal branch.

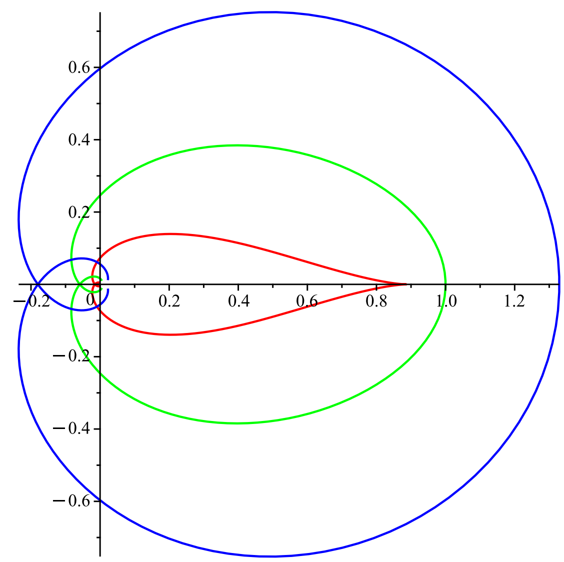

Fig. 5: Contours of Figure 4 after mapping with Γ. Where a contour intersects itself is the edge of the branch of $\check{\Gamma}$ in its range. By magnifying the region around the origin, one can see that the red contour intersects itself and the axis to the left of the origin.

IV. Graphical Representation

In order to obtain a graphical representation of the branches of Γ, we begin by setting up two complex planes: the range of Γ, which is also the domain of $\check{\Gamma}$, and vice versa. On these we mark the intervals applying to the real values of the principal branch. In Figure 3, the top axes are the domain of $\check{\Gamma}$ (and the range of Γ) and since $\check{\Gamma}$ is real for $x \geq \gamma_0 \approx 0.886$, this interval is marked in red on the real axis. The bottom axes are the range of $\check{\Gamma}$ and the real axis is marked with the interval $\check{\Gamma} \geq \psi_0 \approx 1.46$, in which the real values of $\check{\Gamma}$ lie, both sets of markings being in accord with Table II. We now draw contours in the lower axes and map them using Γ to the domain in the top axes.

To move from real values to complex values, we erect contours on the plotted values of $\check{\Gamma}$, as shown in Figure 4. The length of the contours is arbitrary at this stage (they are straight for convenience). We now apply Γ to the contours and obtain the curves in Figure 5 in the domain of $\check{\Gamma}$. We notice that the curves self-intersect. Where they intersect is a discontinuity in function values; in other words a branch cut. The curves after the intersection are violating our requirement that the function is injective. Therefore we return to the straight contours in figure 4 and trim their lengths until their images meet on the real axis without intersecting. This stage of the construction is shown in figures 6 and 7.

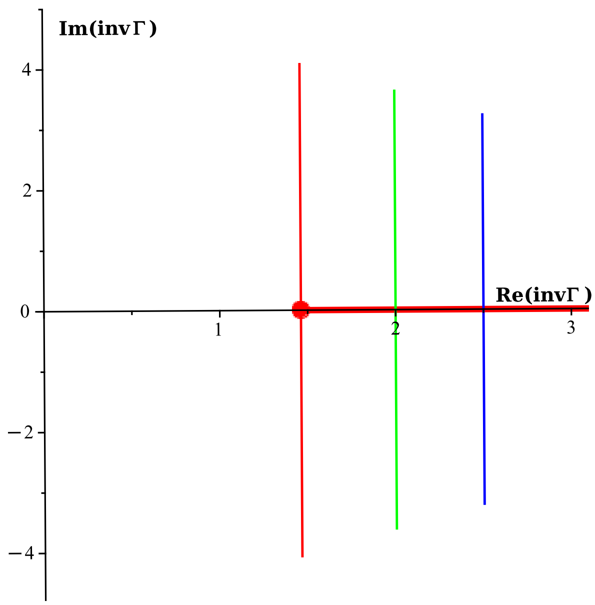

Fig. 6: Corrected contours in $\mathcal{D}_0$ that visualize the range of the $\check{\Gamma}_0$ principal branch.

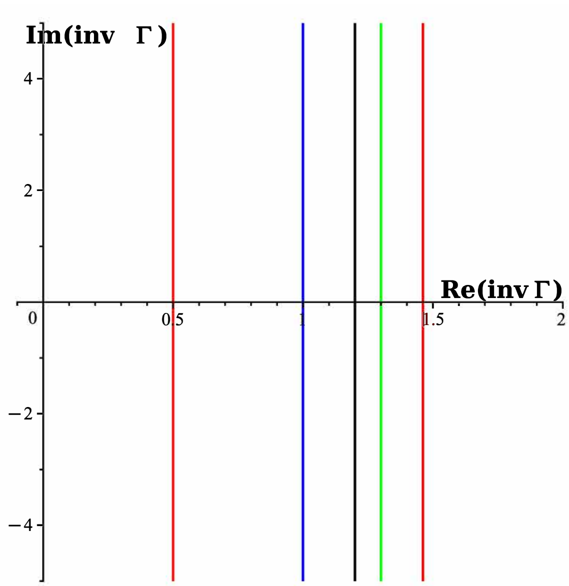

Fig. 8: Exploring the domain of the principal branch by contours which include forbidden parts of the domain of $\check{\Gamma}$ (principal branch).

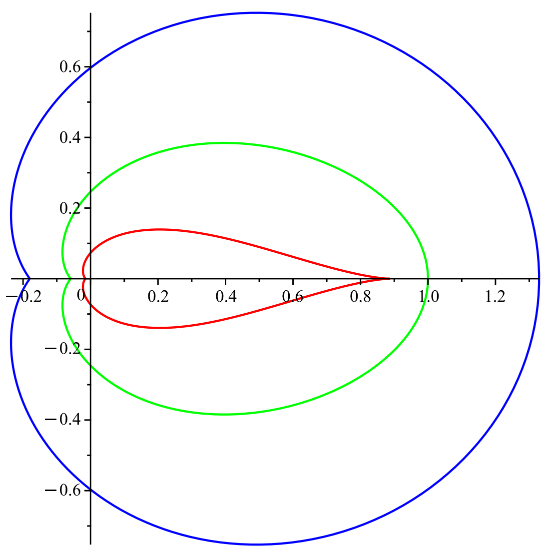

Fig. 7: Corrected contours of Figure 6 after mapping with Γ. The contours no longer intersect and stop at the axis.

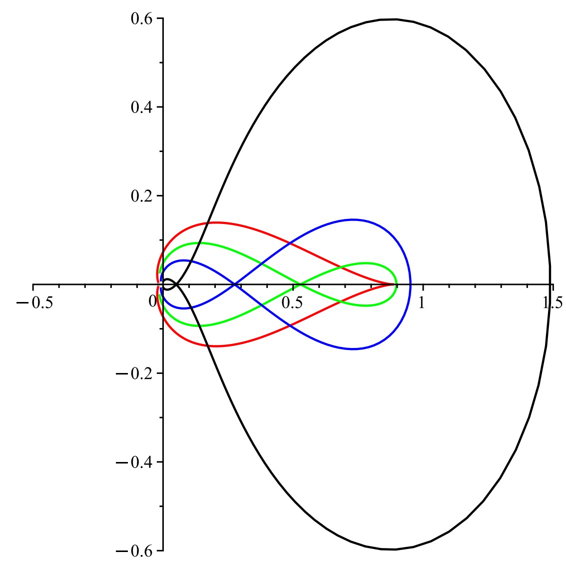

Fig. 9: Contours from figure 8 showing intersections from sections of the contour touching the real axis. The contours in figure 8 must be trimmed both near the axis and at the upper end.

We have now established that additional contours stretching to the right of those shown in figure 6 will account for all of the complex plane outside the contours shown in figure 7. This leaves the region inside the innermost (red) curve unaccounted for. In order to place contours inside the red curve, we have to add contours to Figure 6 and them map them as before; there is, however, a difficulty. The values taken by the principal branch obey $\check{\Gamma} \geq \gamma_0$, and those values have already been used to build contours. Looking at figure 4, we see we cannot use the segment of the real axis that is not marked in red. We can, though, add contours there for exploration. These are shown in figure 8 and their mapping in figure 9. Now the mapped contours self-intersect in two places, indicating that the contours need trimming at two places. The trimmed contours are shown in figure 10, and the resulting inner region of the map is shown in figure 11.

Combining the results above, we obtain the complete description of the range, and hence the definition, of the principal branch; see Figure 12. The construction we have been compiling has now revealed two things. In the domain of $\check{\Gamma}_0$ we have two lines of discontinuity, where we had to avoid self-intersections. In other words, there are two branch cuts: $(-\infty, 0)$ and $(0, \gamma_0)$, as shown in Figure 13.

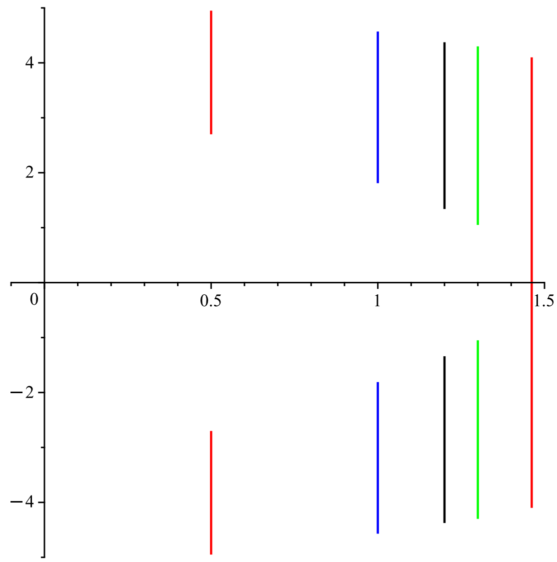

Fig. 10: The contours of figure 8 trimmed to avoid self-intersections.

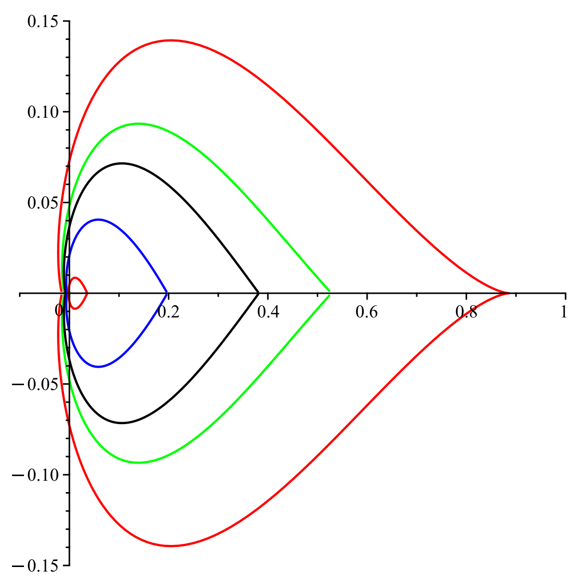

Fig. 11: The contours in figure 10, mapped using Γ. Note that the left sides of the contours intersect the negative real axis.

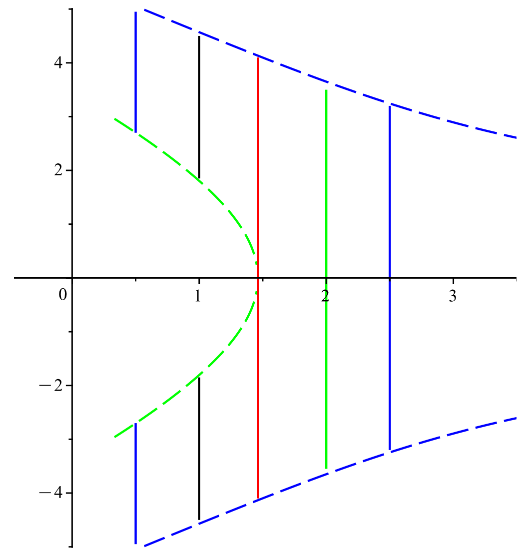

Fig. 12: The range in $\mathcal{D}_0$ of $\overset{\curlyvee}{\Gamma}_0$ as revealed by the contours constructed extending from the real data. The boundary curves (dashed) are obtained from mapping the branch cuts shown in Figure 13 using $\overset{\curlyvee}{\Gamma}$. Note that the colours of the boundaries correspond to the colours of the branch cuts.

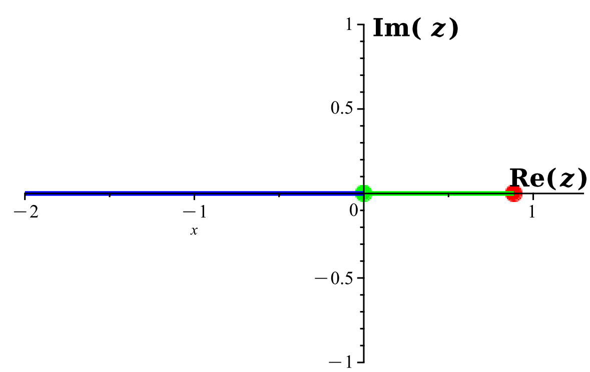

Fig. 13: The branch cuts for the principal branch $\overset{\curlyvee}{\Gamma}_0$. Note that although the two cuts appear to form a continuous line, they are separated by a singularity at the origin.

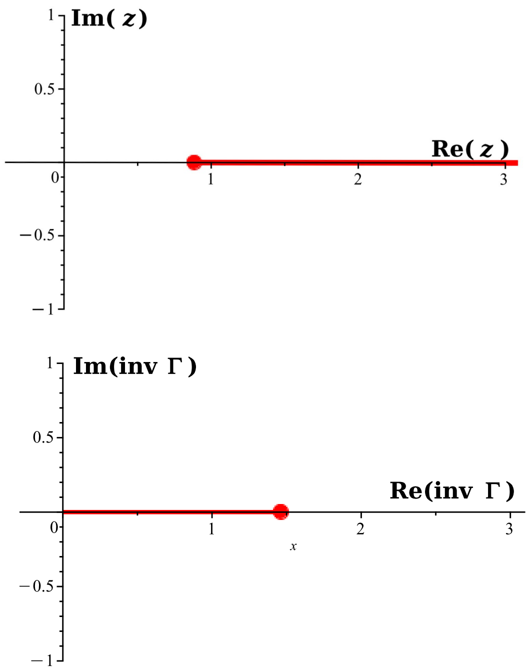

Fig. 14: The domains and ranges for branch $k = -1$. In contrast to $\check{\Gamma}_0$, now $\check{\Gamma}_{-1}$ takes values between 0 and ψ_0.

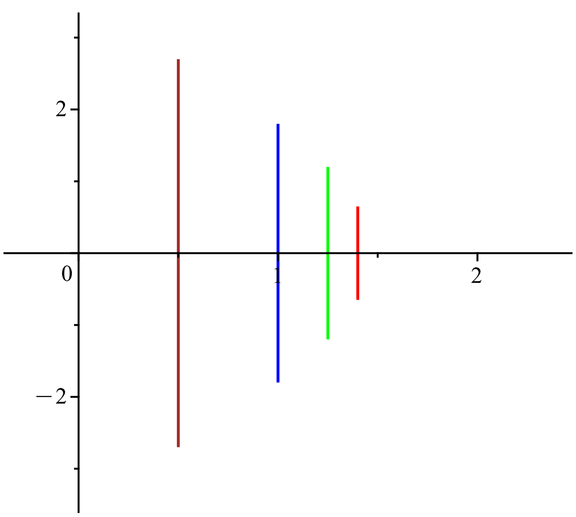

Fig. 15: The range in $\mathcal{D}_{-1}$ of $\check{\Gamma}_{-1}$ as revealed by the contours constructed extending from the real data. The range fits with Figure 12 and the two regions share a boundary.

We can now map the branch cuts into the range of $\check{\Gamma}$ using a crude Maple function for $\check{\Gamma}$, and the result is a plot of the boundaries to the range of $\check{\Gamma}_0$. It should be noted that it is tempting to treat the two branch cuts as one, and simply say that there is a cut $-\infty < x \leq \gamma_0$.

To show that the approach applies to all branches, we briefly consider branch $k = -1$. This branch shares a singular point with branch $k = 0$, namely (ϕ_0, ψ_0), but now values of $\check{\Gamma}_{-1}$ decrease to zero. Thus we replace Figure 3 with Figure 14. Similarly, Figure 12 is replaced by Figure 15, and we note that the two figures fit together like tectonic plates, and share a boundary.

V. Conclusion

In this paper, we have continued to develop the principle that a discussion of the branches of a multivalued function should start in the *range* of the function. This is in contrast to the traditional treatments, for example, in [9], where discussion starts by defining branch cuts in the *domain* of the function (see, for example, [9, Chap. 4] and their sections on logarithm and arctangent). For functions such as $\check{\Gamma}$ or W, where the branch ranges are not regular, the branch cuts in the domain alone are not sufficient for understanding the structure of the branches. It is clear that the inverse Γ function still possesses many interesting properties, which have not been touched yet. These issues are being explored in depth at present.

References

[1] Bradford, R.J. Corless, R.M., Jeffrey, D.J. & Watt, S.M., Reasoning about the Elementary Functions of Complex Analysis, *Annals of Mathematics and Artificial Intilligence* **36** (2002) pp.303-318.

[2] Davenport, J.H., The Challenges of Multivalued "Functions", CICM 2010: Intelligent Computer Mathematics, Springer LNCS 6176, pp 1–12.

[3] Chyzak, F., Davenport, J., Koutschan, Ch. & Salvy, B., On Kahan's Rules for Determining Branch Cuts, 2011 International Symposium on Symbolic and Numeric Algorithms for Scientific Computing, IEEE 2012, pp. 47–51.

[4] England, M., Bradford, R. Davenport, J.H. & Wilson, D., Understanding Branch Cuts of Expressions, Conference on Intelligent Computer Mathematics (CICM 2013), Springer LNCS 7961, pp 136–151.

[5] Borwein, J.M., & Corless, R. M., The Gamma function in the Monthly, American Math Monthly, 125 (5), 400-424, 2018. arXiv:1703.05349 [math.HO]

[6] Pedersen,H.L. Inverses of Gamma functions, Constructive Approximation, **41**, 251–267, 2015.

[7] Folitse, K. A., Jeffrey, D. J. & Corless, R. M., Properties and Computation of the Functional Inverse of Gamma, 19th International Symposium on Symbolic and Numeric Algorithms for Scientific Computing (SYNASC), pp. 63–66, IEEE 2017.

[8] Gronau, D. Why is the Gamma function so as it is, Teaching Mathematics and Computer Science, **1**, 43–53, 2003.

[9] National Institute of Standards and Technology, NIST Digital Library of Mathematical Functions, https://dlmf.nist.gov Retrieved 2023-07-01.

[10] Jeffrey, D.J., Multivalued elementary functions in computer-algebra systems. Proceedings AISC 2014. LNCS, Vol 8884. Ed. Aranda-Corral, GA, Calmet, J, Martin-Mateos,FJ. pp 157–167, Springer 2014.

[11] Jeffrey, D.J. Branch structure and implementation of Lambert W, Mathematics in Computer Science 11, pp 341–350, 2017.

[12] Jeffrey, D.J. & Watt, S.M., Working with Families of Inverse Functions, International Conference on Intelligent Computer Mathematics, pp. 222–237, Springer 2022.

[13] Davis, P.J., Leonhard Euler's Integral: A Historical Profile of the Gamma Function: In Memoriam: Milton Abramowitz. The American Mathematical Monthly, **66**(10), 849-869,

[14] Kahan, W. M., Branch cuts for complex elementary functions, or, Much Ado about Nothing's Sign Bit, In The state of the art in numerical analysis: Proceedings of the Joint IMA/SIAM Conference, Oxford University Press, 1986.