

Working with Families of Inverse Functions

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Abstract. When evaluating or simplifying mathematical expressions, the question arises of how to handle inverse functions. The problem is that for a non-injective function $f : D \rightarrow R$, the inverse is generally not a function $R \rightarrow D$ since there may be multiple pre-images for a given point. The majority of work in this area has fallen into two camps: either the inverse functions, and expressions involving them, are treated as multi-valued objects, or inverse functions are taken to have one principal value. Both these approaches lead to difficulties in evaluation and simplification. It is possible to define the inverse as a function from R to sets of elements of D , but then the algebra of expressions involving the inverse becomes overly complicated. This article extends previous work based on a different approach: instead, the inverse of a function is taken to be a labelled family of functions, with the label specifying the pre-image in the original function's domain. This convention is already used by some authors for logarithms, but it can be applied more generally. In some cases, the branch indices can appear in identities that give more broadly applicable simplification rules. In this paper we survey how this approach can be applied to elementary functions, including the Lambert W function, and give examples using Maple.

Keywords: Inverse functions · Simplification Rules · Branch Cuts.

1 Introduction

Consider the following integral evaluated in Maple.

$$\int \sqrt{2+x}\sqrt{1-x} dx = -\frac{\sqrt{2+x}(1-x)^{\frac{3}{2}}}{2} + \frac{3\sqrt{2+x}\sqrt{1-x}}{4} + \frac{9\sqrt{(1-x)(2+x)} \arcsin\left(\frac{2x}{3} + \frac{1}{3}\right)}{8\sqrt{1-x}\sqrt{2+x}}. \quad (1)$$

The fraction $\sqrt{(1-x)(2+x)}/\sqrt{1-x}\sqrt{2+x}$ in the last term appears unnecessary. Abramowitz & Stegun [1] give the solution of the cubic equation $x^3 + 3px - 2q = 0$ as

$$x = (q + \sqrt{q^2 + p^3})^{\frac{1}{3}} + (q - \sqrt{q^2 + p^3})^{\frac{1}{3}},$$

as did early Maple, but now Maple gives

$$x = \left(q + \sqrt{p^3 + q^2}\right)^{\frac{1}{3}} - \frac{p}{\left(q + \sqrt{p^3 + q^2}\right)^{\frac{1}{3}}}.$$

For $p = -2, q = 1$ the solutions agree, but for $p = 2, q = 1$ they do not. Can a computer system compare the expressions symbolically? The Lambert W function has the unlikely simplification $W_1\left(\frac{1-i\pi}{e}\right) = -1 + i\pi$. We ask how such simplifications can be programmed.

This paper is organized as follows: Section 2 introduces the issues relating to expressions involving inverse functions and some of the literature on the topic. Section 3 develops the ideas of inverse function families for the elementary functions and Section 4 presents some applications. Sections 3 and 4 recapitulate previous work. Section 5 explores these concepts when the function is not periodic and showcases the importance of considering the desired range of the inverse. Section 6 presents two new theorems where the proofs need the handling of inverse functions as described here. Section 7 shows how generalizing branch indices to non-integer values can provide useful pseudoinverses. Finally, Section 8 gives some conclusions.

2 Basic concepts

In mathematical computation, symbolic or numeric, one encounters expressions consisting of nested application of functions to constants and variables drawn from some domains. These expressions may be viewed as functions producing values or as symbolic expressions in a free algebra. These two views are not incompatible — it is often convenient to view expressions as functions that operate on values that are symbolic expressions in a free algebra. In both views we talk about *evaluation* and *simplification*.

When expressions are viewed as functions, evaluation means replacing the variables with values from their domains and applying the composed functions to produce a result. Simplification means producing an equivalent function that is simpler according to some criterion, such as having smaller expression size, lower evaluation cost or better approximation properties. Here the notion of function equivalence is the usual one, meaning that both functions have the same domains and both produce equivalent results for all values in their domain. What is meant by “equivalent results” will depend on the setting.

When viewed as symbolic expressions from a free algebra, evaluation means replacing variables with free algebra terms to produce well-formed expressions. Simplification again means to produce equivalent alternatives that are simpler according to some criterion, usually expression size. Here the notion of equivalence is more subtle, however. When all the free algebra axioms are universally quantified, then expressions are equivalent if each can be transformed to the other by application of the algebraic axioms. When function arguments may represent values

outside their domain, such as fields not allowing division by zero, then equivalence must require that transformations are applied only when the subexpressions for function arguments could never evaluate to excluded values. Alternatively, evaluation may produce exceptions, or simplification may introduce restrictions on the variables. All this can be stated more precisely, but the present explanation suffices for our purposes.

In some settings we may have “multivalued functions”. These have been formulated in the literature in various ways. A well-known example is Carathéodory’s statement [2] that $\ln(AB) = \ln A + \ln B$, which he interprets as each \ln evaluating to a set and that the addition requires selecting values from each set to make the equation true. Selecting other values does not give equality. A second example is $\ln(x^2) = 2 \ln x$. Letting \ln_0 denote the principal branch³, the left-hand side must be the set

$$\{\ln_0(x^2) + 2\pi ik, k \in \mathbb{Z}\}$$

whereas the right-hand side is

$$2 \times \{\ln_0 x + 2\pi ik, k \in \mathbb{Z}\} = \{2 \ln_0 x + 4\pi ik, k \in \mathbb{Z}\}.$$

In working with expressions, the domain of the function arguments and variables is essential. We focus on the situation in which there is one domain. That is, for some domain D , all functions have signature $D^n \rightarrow D$ for some n . Multivalued functions are then functions mapping to $D^m, m > 1$. This situation arises naturally when expressions include non-injective functions and it is desired to construct expressions involving their inverses. Many authors allow such functions to occur in expressions, implying that the expressions now represent values in $D^* = \bigcup_{i=0}^{\infty} D^i$. This approach has several problems, including that the resulting sets may contain extraneous or infeasible points and that the usual axioms used for simplification may no longer be valid.

The problem of multivalued functions cannot be ignored, however, as it remains necessary to treat expressions with function inverses.

In general, a function $f : D \rightarrow D$ will have an inverse $\text{inv}(f) : f(D) \subseteq D \rightarrow D$ only if f is injective (1-1). For non-injective functions, we may partition D as $D = \bigcup_{i=1}^n D_i$, where $D_i \cap D_j = \emptyset$ if $i \neq j$, and such that $f|_{D_i} : D_i \rightarrow D$ is injective for each i , introducing a *family of inverse functions* $\text{inv}(f)_i = \text{inv}(f|_{D_i})$. The choice of partitioning is not generally unique, and if D is a metric space, then it is usual to take D_i as connected components and to call $f(\partial D_i)$, the images of the partition boundaries, *branch cuts*. Even if f is continuous on D then $\text{inv}(f)_i$ will in general not be continuous on the branch cuts. Viewing the graphs functions $D \rightarrow D$ as subsets of $D \times D$, we may interpret $\bigcup_i \text{inv}(f)_i$ as a Riemann surface.

The approach of using a family of inverse functions, as opposed to a single multivalued function, is that it becomes possible to write identities that

³ When different authors use the same symbol to mean different things, the discussion of notation becomes problematic. For this example, \ln is a set; \ln_0 is a unique value. Below, we shall change to notation in which \ln is also a unique value.

hold over larger domains. For example, with $\text{inv}(\exp)_k z = \ln z + 2k\pi i$,

$$\begin{aligned} \text{inv}(\exp)_m A + \text{inv}(\exp)_n B &= \text{inv}(\exp)_s(AB) , \\ s &= m + n + \mathcal{K}(\text{inv}(\exp)_0 A + \text{inv}(\exp)_0 B) , \end{aligned}$$

where \mathcal{K} is the unwinding number [3] and \ln is the usual principal branch. The implementation of multivalued functions in computational environments, whether numerical (e.g. Matlab) or symbolic (Maple, Mathematica, Sage, etc), has been a topic of ongoing discussions between mathematicians and system implementers [2, 3, 8, 5]. One can say that mathematicians prefer an *ad hoc* interactive approach in which a function evaluation would return a set of possible values, which the mathematician can employ as the problem demands. In contrast, computer systems require deterministic rules which predetermine what will be returned by a function evaluation. This dichotomy has been adequately discussed in the above references, and is not repeated here.

Some readers may point out that there are examples of computer systems being ambiguous in the meaning of a multivalued function, similar to mathematicians. In Maple, the help page for `RootOf` describes it as a placeholder for all the roots of an equation in one variable. In contrast `evalf(RootOf(x2 - 4))` evaluates to 2 only. This special case, however, distracts from the main ideas here, and will not be pursued further.

One theme of this review is the roles played by the function domain and range in the understanding of multivalued functions. Given a function $f(z)$, earlier discussions have focused on the domain of f , and detailed the branch cuts present in the domain. For example, the DLMF [11] defines complex logarithm, and complex inverse trigonometric functions showing diagrams of their domains containing branch cuts, but no diagrams of the ranges. The view here is that the ranges of these functions show the nature of the multi-valuedness more clearly than the domain alone.

3 The elementary functions

The basis for the discussion is an extension of the notation for the natural logarithm that was introduced in [4]. We first consider each of the elementary functions in pairs, a function and its inverse, using this approach.

3.1 Exponential and logarithm

We begin by declaring that $\ln z$ denotes the uniquely defined principal branch function seen in all computational systems, numerical or symbolic. Figure 1 shows the domain and range of $\ln z$. The solid red line denotes the branch cut in the domain, and its image in the range. We note that if $z = x + iy$ then the branch cut is $\{z = x \mid -\infty < x < 0\}$ and its image in the range is $\{\ln(-x) + i\pi \mid -\infty < x < 0\}$. The dashed green line in the range is the line $\{\ln(-x) - i\pi \mid -\infty < x < 0\}$ and marks

the lower boundary of the range, but does not belong to the range. It is important to note that the values taken by this logarithm are confined to the region between the red and green lines. Points outside this region cannot be reached.

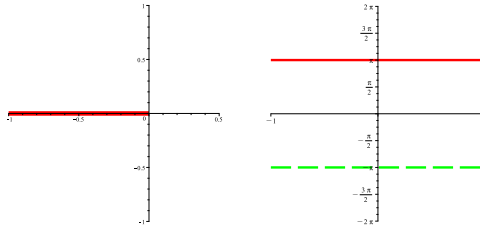


Fig. 1. Domain (left) and range (right) of the principal branch logarithm. The range is confined between the horizontal lines.

This is the motivation for defining a family of logarithm functions which together allow all points in the range to be reached. Using a subscript $k \in \mathbb{Z}$, we have:

$$\ln_k z = \ln z + 2\pi i k .$$

Although k is here an integer, there are possibilities for non-integer values being useful. In order to cover the range completely and without overlap, however, integers are necessary. We can describe $\ln_k z$ as the k th branch of logarithm. An immediate benefit of this notation is a precise statement regarding a well-known property of logarithm, which otherwise is justified ambiguously by Carathéodory [2]. We state

$$\forall k \in \mathbb{Z}, \exists m, n \in \mathbb{Z}, \text{ such that } \ln_k z_1 z_2 = \ln_m z_1 + \ln_n z_2 .$$

An application of the definition allows us to describe compactly the asymptotic behaviour of the Lambert W function.

$$W_k(z) \approx \ln_k(z) - \ln \ln_k(z) ,$$

where it should be noted that two different branches of logarithm are used. A Maple implementation of the extended function is

```
Ln := proc (z::algebraic) local branch;
  if nargs <> 1 then
    error "Expecting 1 argument, got", nargs
  elif type(procname, 'indexed') then
    branch := op(procname); ln(z) + 2*Pi*I*branch;
  else ln(z)
  end if
end proc;
```

Maple's names for the function, `log` and `ln` are left unchanged and the new name `Ln` created. Note that the name `Log` is not used, because there could be confusion with the subscript indicating the base of the logarithm. Thus `Ln[2](5.) = 1.60944+12.566 i` .

3.2 Sine and arcsine

As with logarithm, the range of the arcsine function is confined to a strip in the complex plane, in this case parallel to the imaginary axis, and the remainder of the range cannot be reached by the function. Branch cuts in the domain of arcsine are $\{z = x \mid -\infty < x < -1 \cup 1 < x < \infty\}$. In the range, the vertical lines are $\{\sin(\pm\pi/2 + iy) \mid -\infty < y < \infty\}$ with the solid red boundaries belonging to the range of the function and the dashed green parts showing the boundary without being part of the range. As with logarithm, the only way to reach the parts of the range outside the principal strip is by defining additional branches of arcsine (Figure 2) As with logarithm, the only way to reach the parts of the

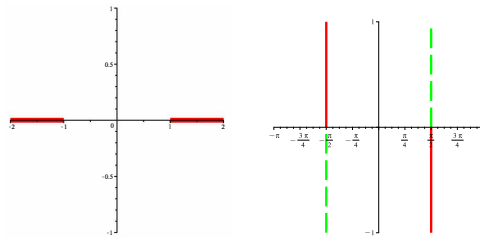


Fig. 2. Domain and range of arcsine. The solid red lines in the range correspond to the solid red lines in the domain. The dashed green lines mark the border of the range, but do not belong to it.

range outside the principal strip is by defining additional branches of arcsine. In order to avoid clashes between Maple's standard notation of `arcsin`, the extensions here, are denoted `invsin`. This name is modeled on Maple's `invfunc` notation, but the use differs in that `invfunc` is a table, whereas we use `invsin` as a function. Using this, we extend the definition to a branched inverse sine by

$$\text{invsin}_0 z = \arcsin z , \tag{2}$$

$$\text{invsin}_k z = (-1)^k \text{invsin}_0 z + k\pi . \tag{3}$$

The principal branch now has the equivalent representation $\text{invsin}_0 z = \arcsin z$. It has real part between $-\pi/2$ and $\pi/2$. Notice that the branches are spaced a distance π apart in accordance with the an-

tipperiod⁴ of sine, but the repeating unit is of length 2π in accord with the period of sine.

The Maple code for the function is

```
invsin := proc (z::algebraic) local branch;
    if nargs <> 1 then
        error "Expecting 1 argument, got", nargs
    elif type(procname, 'indexed') then
        branch := op(procname);
        branch*Pi+(-1)^branch*arcsin(z)
    else arcsin(z)
    end if
end proc;
```

Examples of its use appear below.

3.3 Inverse cosine

The shift from inverse sine to inverse cosine is straightforward. The principal branch has real part between 0 and π , and this is easiest achieved by setting $\text{invcos}_k z = \text{invsin}_{k+1} z - \pi/2$. The code is

```
invcos := proc (z::algebraic) local branch;
    if nargs <> 1 then
        error "Expecting 1 argument, got", nargs
    elif type(procname, 'indexed') then
        branch := op(procname);
        invsin[branch+1](z)-Pi/2
    else arccos(z)
    end if
end proc;
```

3.4 Inverse tangent

The principal branch has real part from $-\pi/2$ to $\pi/2$, and the k th branch is $\text{invtan}_k z = \text{invtan} z + k\pi$. As code:

⁴ An antiperiodic function is one for which $\exists\alpha$ such that $f(z + \alpha) = -f(z)$, and α is then the antiperiod. This is a special case of a quasi-periodic function [10], namely one for which $\exists\alpha, \beta$ such that $f(z + \alpha) = \beta f(z)$.

```

invtan := proc (z::algebraic) local branch;
    if nargs <> 1 then
        error "Expecting 1 argument, got", nargs
    elif type(procname, 'indexed') then
        branch := op(procname); branch*Pi+arctan(z)
    else arctan(z)
    end if
end proc;

```

The two-argument inverse tangent function has been implemented in many computer languages. It is a synonym for `arg`, meaning the argument or phase of a complex number, in that $\arg(x + iy) = \arctan(y, x)$ for $x, y \in \mathbb{R}$. It can be described using the branches of `invtan` as

$$\arctan(y, x) = \operatorname{invtan}_k(y/x) ,$$

where $k = H(-x) \operatorname{sgn} y$, and H is the Heaviside step function.

3.5 Fractional powers

The principal branch of $z^{1/n}$ is defined by $\exp(\frac{1}{n} \ln z)$, and replacing $\ln z$ by $\ln_k z$ gives the branched function. The standard notation for roots and fractional powers does not leave an obvious place for the branch label, and most obvious names are already used by Maple or Mathematica. We use the name `invpw`, meaning inverse (integer) power. The Maple code defines `invpw[k](z,n)`, where the subscript is the branch, as usual, while the fractional power is $1/n$. Thus it is modelled on the Maple `surd` function. Unlike the other inverse functions, there are only n distinct values, but we allow k to be any integer.

Since square root is so common, it is coded separately as `invsq[k](z)`, and it can be displayed in traditional notation as $(-1)^k \sqrt{z}$.

3.6 Definitions in terms of logarithms

Kahan[9] prefers to give definitions of the inverse trigonometric functions in terms of logarithms. We may extend these definitions to the families of inverse functions as follows:

$$\begin{aligned}
 \operatorname{invsh}_k z &= (-1)^k \ln_{(-1)^k \frac{k}{2}} \left(z + \sqrt{1 + z^2} \right) & \operatorname{invsin}_k z &= \frac{1}{i} \operatorname{invsh}_k iz \\
 \operatorname{invch}_k z &= \frac{1}{i} \operatorname{invcos}_k z & \operatorname{invcos}_k z &= \frac{\pi}{2} - \operatorname{invsin}_{-k} z \\
 \operatorname{invth}_k z &= \frac{1}{2} \left(\ln_{\frac{k}{2}}(1 + z) - \ln_{-\frac{k}{2}}(1 - z) \right) & \operatorname{invtan}_k z &= \frac{1}{i} \operatorname{invth}_k iz
 \end{aligned}$$

These cover the domains of the functions inverted. The details of how to ensure continuity on a path between branches is left for a future article.

4 Applications

We now demonstrate some uses of the new notation.

4.1 Plotting

With the new functions, we can easily plot branches. Figure 3 shows plots produced by the Maple commands

```
> plot([invsin[-1](x),invsin(x),invsin[1](x)],x=-1 .. 1,
        linestyle=[dot,solid,dash]);
> plot([invtan[-1](x),invtan(x),invtan[1](x),invtan[2](x)],
        x=-5..5,linestyle=[dot,solid,dash,dashdot]);
```

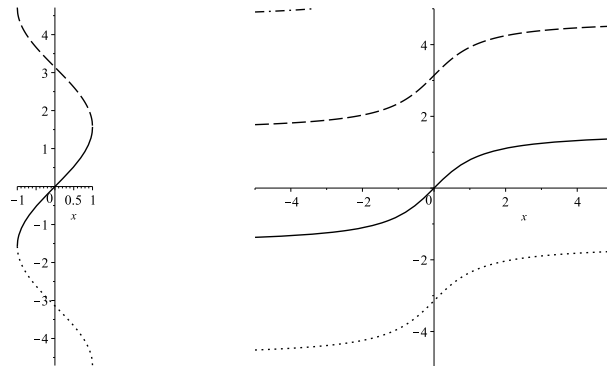


Fig. 3. The branches of inverse sine and inverse tangent plotted taking advantage of branch notation.

4.2 Identities

In order to express identities containing inverse functions correctly, we need the unwinding number,

$$\mathcal{K}(z) = \left\lceil \frac{z - \pi}{2\pi} \right\rceil ,$$

defined in [3] (rather than in [4] where the sign is different). Note that the unwinding number is a built-in function in Maple, called `unwindK`. This immediately gives us

$$\ln_k e^z = z - 2\pi i \mathcal{K}(z) + 2\pi i k . \quad (4)$$

Note the special case $\ln_{\mathcal{K}(z)} e^z = z$.

Consider an identity one might see in a traditional treatment:

$$\cos x = \sqrt{1 - \sin^2 x} , \quad (5)$$

where the author would add “and the branch of the root is chosen appropriately”. Using the branched root, we write the more precise

$$\cos x = \text{invsq}[\mathcal{K}(2ix)](1 - \sin^2 x) = (-1)^{\mathcal{K}(2ix)} \sqrt{1 - \sin^2 x} . \quad (6)$$

We can contrast the two approaches in Maple with the command

```
> plot([ sqrt(1-sin(x)^2), invsq[unwindK(2*x*I)](1-sin(x)^2)],
      x = -7 .. 7, linestyle = [dot,solid]);
```

The resulting plot is given in figure 4.

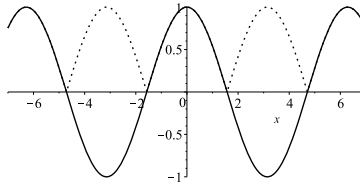


Fig. 4. The graph of $\sqrt{1 - \sin^2 x}$ using branch notation for square root. The dotted line uses an unbranched sqrt, meaning it is plotted using the built-in sqrt, which leaves sign choices to the user; the solid line uses a sqrt function that allows the sign choice to be part of the definition, rather than a separately added sign.

The Abramowitz and Stegun [1] ‘identity’ for adding arctangents is

$$\text{Arctan } x + \text{Arctan } y = \text{Arctan } \frac{x + y}{1 - xy}$$

The more precise identity is

$$\text{invtan}(x) + \text{invtan}(y) = \text{invtan}_k \frac{x + y}{1 - xy}, \text{ where } k = H(xy - 1) \text{sgn}(x) , \quad (7)$$

and H is the Heaviside step. A more complicated example from [1] is their identity for $\text{Arcsin } x + \text{Arcsin } y$, which becomes

$$\begin{aligned} \text{invsin } x + \text{invsin } y &= \text{invsin}[k] \left(x\sqrt{1 - y^2} + y\sqrt{1 - x^2} \right) , \quad (8) \\ k &= H(x^2 + y^2 - 1)(\text{sgn } x + \text{sgn } y)/2 . \end{aligned}$$

Here the branch of invsin is allowed to vary, but there might be another formula which includes variable branches of square root.

As a final identity, we consider formula (4.4.39) in [1].

$$\text{Arctan}(x + iy) = k\pi + \frac{1}{2} \arctan \frac{2x}{1 - x^2 - y^2} + \frac{i}{4} \ln \frac{x^2 + (y + 1)^2}{x^2 + (y - 1)^2} .$$

To turn this identity into something that computer-algebra systems can use, one should decide what to do with k . This can be replaced by

$$\text{invtan}_k(x + iy) = \frac{1}{2} \text{invtan}_n \frac{2x}{1 - x^2 - y^2} + \frac{i}{4} \ln \frac{x^2 + (y + 1)^2}{x^2 + (y - 1)^2} ,$$

where $n = 2k + \text{sgn}(x)H(x^2 + y^2 - 1)$.

4.3 Calculus

Calculating the derivative of an inverse function is a standard topic in calculus. The results in the textbooks are restricted to the principal branches of the functions. It is possible, however, to generalize results to any branch. For example

$$\frac{d}{dx} \text{invsin}_k x = \frac{1}{\cos(\text{invsin}_k x)} = \frac{(-1)^k}{\sqrt{1 - x^2}} .$$

Integration by substitution is a well-known application of inverse functions. A specific difficulty has been the application of the substitution $u = \tan \frac{1}{2}x$ in integrals such as

$$\int \frac{3 dx}{5 - 4 \cos x} = \int \frac{6 du}{1 + 9u^2} = 2 \arctan(3 \tan \frac{1}{2}x) . \quad (9)$$

The right-hand side is discontinuous, as has been pointed out in [7, 6]. The correction to the usual integration formula [7] can be rewritten in the new notation as

$$\int \frac{3 dx}{5 - 4 \cos x} = 2 \text{invtan}_{\mathcal{K}(ix)}(3 \tan \frac{1}{2}x) . \quad (10)$$

The contrast is illustrated in figure 5 by the plot

```
> plot([ 2*invtan[unwindK(I*x)](3*tan((1/2)*x)),
        2*arctan(3*tan((1/2)*x))], x=-3..9, linestyle=[dot,solid],
        discont=true);
```

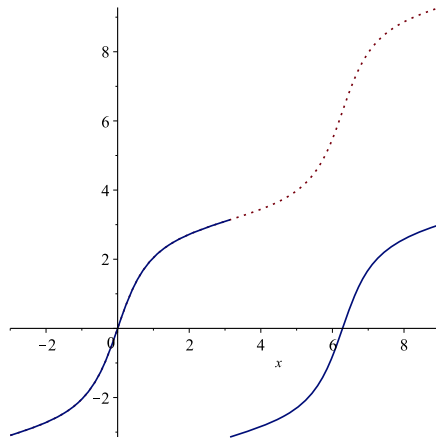


Fig. 5. A graph of the discontinuous and continuous integral expressions. The solid curves are the discontinuous expression (9). The dotted curve follows (10), which coincides with (9) for $x < \pi$, but then extends the integral continuously. The benefits of the dotted curve are discussed in [7].

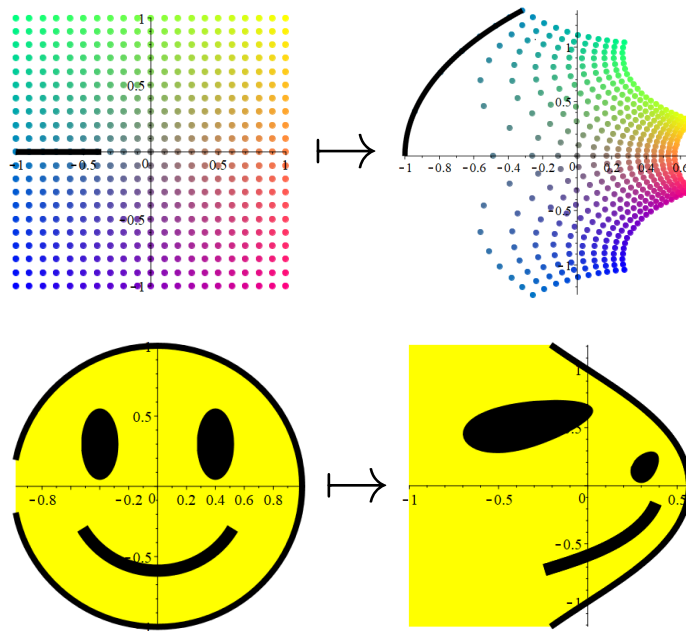


Fig. 6. The W map of the complex plane. The line segment in the top example has domain $x \in [-1, -1/\epsilon]$. The range of the second example is clipped.

5 Simplification of Lambert W

An important point to remember when working with branches is that the relevant branch is determined by the value of the function or expression, more than by the value of the argument. This means working and thinking in the range of the expression, rather than trying to follow the effects of branch cuts in the domain. Here we use this to simplify the Lambert W function for special values. We recall that the Lambert W function is defined by

$$W(z)e^{W(z)} = z, \quad z \in \mathbb{C}. \quad (11)$$

The principal branch is illustrated in Figure 6. The domain contains the branch cut $z = \{(x, 0) \mid x \leq -1/e\}$.

The simplification question is as follows: Consider $x \in \mathbb{R}$, and ask when $W = x - i\pi$, with $\in \mathbb{C}$. That is, we ask about the behaviour of W on horizontal lines, as shown in figure 7 We have

$$We^W = (x - i\pi)e^{x-i\pi} = e^x(-x + i\pi).$$

More generally,

$$(x + \alpha i\pi)e^{x+\alpha i\pi} = e^x(x + \alpha i\pi)e^{\alpha \pi i},$$

and for $\alpha = n/2$ with $n \in \mathbb{Z}$ we get a simplification of the exponential term. The difficulty is identifying the branch to which things apply. This is determined by the value of W , not by its argument. So although the e^x attracts attention, it does not affect the branch.

Since $x + \alpha i\pi$ is a horizontal line, we can start with $x = 0$.

For $x = 0, \alpha = 1/2$, that point is where the branch boundary crosses the axis. So for $x < 0$ we need branch=1, and for $x \geq 0$ we need branch 0.

For $x = 0, \alpha = 1, 3/2, 2$ the points lie in branch=1 and the line stays in branch 1 for all x .

So for $\alpha = (4m + 1)/2$ we have the rule

```

alph:=(4*m+1)/2;
if m>=0 and x>=0 then k:=m end if;
if m>=0 and x<0 then k:=m+1 end if;
if m<0 then k:=m end if;
W(k,exp(x)*(x*I-alph*Pi))=x+I*Pi*alph;

```

Note that for $m < 0$ we do not cross boundary.

Now if we set $\alpha = (4m - 1)/2$, we cross boundaries below.

```

alph:=(4*m-1)/2;
if m<=0 then if x>=0 then k:=m; else k:=m-1 end if;
else      k:=m
end if;
W(k,exp(x)*(-x*I+alph*Pi))=x+I*Pi*alph;

```

There will be similar rules for $\alpha = (2m \pm 1)/2$, but without the x check. Also, if we substitute $\ln x$ for x , we get

$$W(k, x(\ln x + \alpha i\pi))e^{i\alpha\pi} = \ln x + \alpha i\pi$$

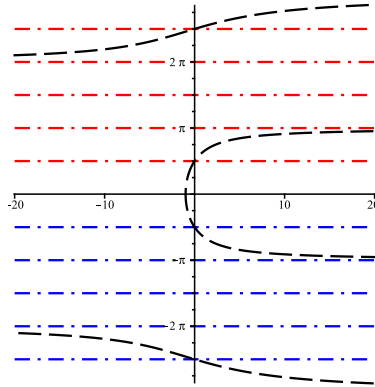


Fig. 7. The range of W showing the branches and the simplification contours. The black curves are the branch boundaries; the red and blue lines $x + \alpha i\pi$

6 Proof with multivalued functions

Multivalued inverse functions require new ways of thinking of relations. For example, the following theorem is new.

Theorem 1. For all $a, b \in \mathbb{R}$ and for all $z \in \mathbb{C} \setminus \mathbb{R}$, we have

$$\sqrt{a+z}\sqrt{b-z} = \sqrt{(a+z)(b-z)}.$$

Proof Without loss of generality, we can assume that z lies in the upper half plane, i.e., we assume that $\Im z > 0$, as shown in figure 8. For all a , $0 < \arg(a+z) < \pi$. Therefore $0 < \arg(\sqrt{a+z}) < \pi/2$. For all b , $-\pi < \arg(b-z) < 0$. Therefore $-\pi/2 < \arg(\sqrt{b-z}) < 0$. Therefore

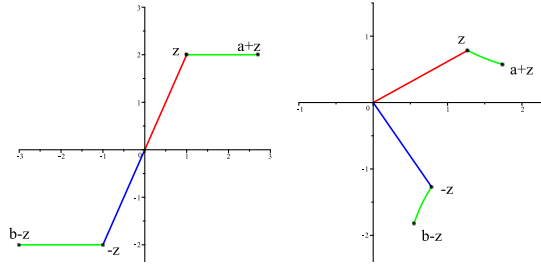


Fig. 8. The imaginary part of z is assumed positive, and $a > 0$ and $b < 0$. Thus $a + z$ is a point to the right of z and $b - z$ is to the left. After mapping under square root, the points both lie in the principal range, and their product also lies in the principal range.

$-\pi/2 < \sqrt{a+z}\sqrt{b-z} < \pi/2$. Also $-\pi < \arg((a+z)(b-z)) < \pi$, and therefore $-\pi/2 < \sqrt{(a+z)(b-z)} < \pi/2$.

An application of this theorem is to integration in Maple. The integral of $\sqrt{2+x}\sqrt{1-x}$ was presented in (1). After applying the theorem, we obtain

$$\int \sqrt{2+x}\sqrt{1-x} dx = \sqrt{2+x}\sqrt{1-x} \left(\frac{1}{4} + \frac{x}{2} \right) + \frac{9}{8} \arcsin\left(\frac{2x}{3} + \frac{1}{3}\right)$$

7 Pseudoinverses

Altering branch structures using rational indices can result in useful pseudoinverses. For example

$$\exp(\ln_{-1/2}(z)) = \exp(\ln z - i\pi) = -z .$$

Here the subscript labels the $-1/2$ branch of the natural logarithm and should not be confused with a logarithm base.

This is what is needed to express the asymptotic behaviour of the Lambert W function in the neighbourhood of the origin. Since the defining equation is

$$W(z) \exp(W(z)) = z ,$$

it is clear that for large z , we need $W(z)$ is large. In fact $W_k(z) \approx \ln_k z$. In addition, however, it is possible that W can become large and negative when z is small. This can be obtained if $\exp(W(z)) \rightarrow -\infty$. Thus we consider the possibility of

$$W(z) \approx \ln_{-1/2}(z) + v .$$

Substituting into the defining equation, and abbreviating $L(z) = \ln_{-1/2}(z) = \ln z - i\pi$, we obtain

$$W \exp(W) = (L(z) + v) \exp(L(z) + v) = z$$

This gives $(L(z) + v)(-z)e^v = z$. This can be simplified to

$$-L(z) - v = e^{-v} .$$

8 Conclusions

We have reviewed the usual approaches to handling multivalued functions, and inverse functions in particular, and have detailed an alternative approach that allows clear treatment of branches and the relations among them. A particular feature of our approach is to emphasize the need to place more emphasis on the range of the inverse function, because it is in the range that the branches are defined. Rather than treating the multivalued inverses as awkward artifacts that make expressions ambiguous, we are able to make precise statements, allow identities to have maximal extent, and do not need *ad hoc* interpretation.

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