A Review of Algorithms for Symbolic Domains

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***Abstract*—This talk is about computing with mathematical quantities where the sizes or shapes are not known in advance. We consider polynomials where the exponents can be given by symbolic expressions, matrices with blocks or other internal structure of symbolic size, and piece-wise functions where the shapes of the domains are given by symbolic expressions.**

We begin with the observation that recent research has focussed on algorithms for *computer algebra*, relying on base operations in an algebraic domain, and less attention has been devoted to *symbolic computation*, where expressions are represented in a term algebra and the form of the expression is meaningful. Some of these ideas are explored in [1], [2].

The first area we review is that of polynomials with symbolic exponents. For such "polynomials" there are various straightforward operations, such as squaring $x^{2n} - 1$ to get $x^{4n} - 2x^{2n} + 1$, or differentiating to get $2nx^{2n-1}$. We review algorithms to compute more sophisticated operations such as the GCD, factorization and functional decomposition of such polynomials. The development of this topic may be found in articles [2], [3], [4], [5].

The second area we review is that of matrices with symbolic structure, that is matrices with blocks, bands and other structures where the dimensions are given by symbolic expressions. We trace the development of some ideas on how how to do arithmetic on these objects. For details see [6], [7], [8]. This work relies on support functions in the first instance built out of steps

$$\sigma(x, y) = \begin{cases} 1 & \textbf{if} \quad x \le y \\ 0 & \textbf{otherwise} \end{cases}$$

and then simplified using oriented intervals

$$\xi_{i,y,z} = \begin{cases} 1 & \textbf{if} \quad y \le i < z \\ -1 & \textbf{if} \quad z \le i < y \\ 0 & \textbf{otherwise}. \end{cases}$$

The third topic generalizes the work on symbolic matrices by recognizing that the use of signs in the oriented intervals ξ above, and in other settings, such as integration

$$\int_a^b = -\int_b^a = \int_a^c + \int_c^b,$$

is at its foundation a manner of including or excluding regions from a domain. We consider the case of piece-wise functions, where the regions of definition are given symbolically. We show how hybrid sets, a generalization of multi-sets allowing negative multiplicities, can be used to describe a generalized notion of inclusion/exclusion, where the exclusion can occur before inclusion. This allows linear algebra on generalized partitions to reduce the computational complexity of working with symbolic piecewise objects from exponential to linear in the number of pieces, and also leads to a particularly elegant formulation of Stokes' theorem [9], [10]. We also observe that the use of hybrid sets leads to interesting combinatorial identities [11].

***Keywords*-computer algebra, symbolic computation, polynomial algebra, matrix algebra, piecewise functions**

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