Introduction to Reconfiguration

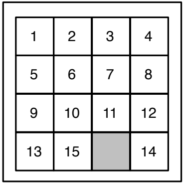
Naomi Nishimura
David R. Cheriton School of Computer Science
University of Waterloo

What do these have in common?

Can you change the word DOG into the word CAT?

DOG
HOG
HAG
HAT
CAT

Can you slide the tiles to put the numbers in order?



Reconfiguration

The **reconfiguration framework** is defined in terms of:

- a classical problem P,
- an instance I of P,
- a definition of **feasibility** of solutions to I, and
- a definition of **adjacency** of feasible solutions.

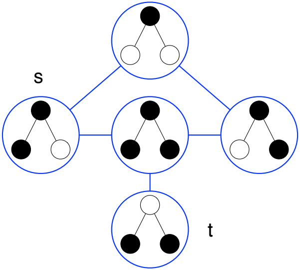
To make the problem interesting/tractable:

- Adjacency is polynomially testable.
- Feasibility is restricted.

Graph representation

A **reconfiguration graph** is defined as

- vertices that represent feasible solutions and
- edges that represent adjacency.



Viewed as a graph	**Viewed as a process**
Vertex	Solution
Edge	**Reconfiguration step**
Path	**Reconfiguration sequence**

Common types of reconfiguration problems

Intertwined structural and algorithmic problems:

Reachability Is there a reconfiguration sequence between the two input solutions?

Connectivity Is the reconfiguration graph connected?

Diameter What is the diameter of the reconfiguration graph?

Shortest transformation What is the shortest reconfiguration sequence between the two input solutions?

Examples revisited

Word reconfiguration

Solution: k-letter word in English

Step: Change one letter

Reachability example: Can you change the word DOG into the word CAT?

15 puzzle reconfiguration

Solution: Arrangement of tiles

Step: Slide one tile

Connectivity example: Are all arrangements reachable from each other?

Warnings and apologies

Goals:

- Whirlwind tour
- Slides on my website to read at leisure

Too little:

- Omissions of definitions
- Omissions of references
- Omissions of related work

Too much:

- Too many slides
- Too many words per slide

Sliding block puzzles

Key idea: Studied in great detail for many years.
Generalize to moving tokens or pebbles on a graph:

Connectivity no for 15-puzzle [Johnson and Story 1879]; yes except if the puzzle graph is a cycle on $n \geq 4$ vertices, is bipartite and not a cycle, or an exceptional graph on vertices [Wilson 1974]

Diameter cubic [Kornhauser, Miller, Spirakis 1984]

Shortest transformation NP-hard [Goldreich 1984]; P when all tokens are the same; NP-hard when at least one token is special [van den Heuvel and Trakultriapruk 2014]

Related work on robot motion planning
[Papadimitriou, Raghavan, Sudan, Tamaki 1994]
multi-colour pebbles [Goraly and Hassin 2008], many others

An incomplete history of reconfiguration

Before 2011 Lots of work not called reconfiguration

2011 Reconfiguration framework [Ito, Demaine, Harvey, Papadimitriou, Sideri, Uehara, Uno 2011]

- Reachability
- PSPACE-hardness results for some NP-hard classical problems and machinery (NCL)
- Polynomial-time algorithms for some polynomially-solvable classical problems
- Conjecture that reachability is not in P for all classical problems in P

Since 2011 Lots of work called reconfiguration (and also lots of work not called configuration)

Applications and related frameworks

Example applications

- Maintaining a firewall in a changing network
- Assigning frequencies in a changing mobile network
- Planning motion, including 3D printing and robot movement
- Glauber dynamics Markov chain from statistical physics

Related frameworks

Local search: Given a solution to an instance, find a better solution that is “close” to the input solution.
Reoptimization: Given an instance, an optimal solution, and changes to the instance, find an optimal solution to the changed instance.
Incremental problems: Given a yes-instance, a witness that it is a yes-instance and changes to the instance, determine if the changed instance is a yes-instance.

Related problems

String, graph, and tree editing Transformation of one entity into another using a fixed set of editing operations

Tree rotation Transformation of a tree into another tree by rotations

Morphing graph drawings Continuous transformation of one shape into another

Linkage reconfiguration Continuous transformation of a simple planar polygon to make it convex - Carpenter's Rule Theorem [Connelly, Demaine, Rote 2003]

Unfolding polyhedra Cutting and flattening polyhedra

Countless others

Outline of the talk

Coverage of:

- Progress on individual problems
- Progress on general properties of the framework

Structured as:

- Reachability
- Connectivity
  - Case study: Dominating set
  - Case study: Colouring
- Diameter
  - Case study: Colouring
  - Parameterizing by length of sequence
  - Case study: Vertex cover
- Shortest transformation
  - Case study: Flip distance
  - Case study: Satisfiability
- Other research directions

Relating classical and reachability complexity

General pattern

Classical problem NP-hard $\Rightarrow$ reachability PSPACE-hard
Classical problem in P $\Rightarrow$ reachability in P

- PSPACE-completeness of Independent Set, Clique, Vertex Cover, Set Cover,Dominating Set, Integer Programming, Vertex Colouring, List Edge-Colouring, Satisfiability, Steiner Tree
- Polynomial-time algorithms for Minimum Spanning Tree and Matching

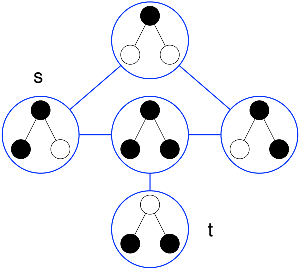
Conjecture [Ito et al. 2011]

Not all classical problems in P give rise to polynomially-solvable reconfiguration problems.

Common types of reconfiguration steps

Graph problems Solutions are sets of vertices, viewed as marked by tokens.

- **Token Jumping** (TJ) (all solutions are the same size)
- **Token Sliding** (TS) (all solutions are the same size)
- **Token Addition and Removal** (TAR)
  (a bound on size is usually required to avoid triviality)



Satisfiability Solutions are truth assignments.
Change one variable from true to false or from false to true.

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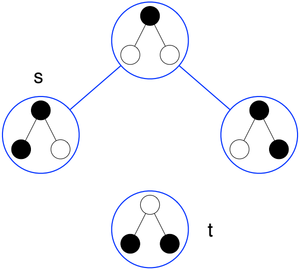


Satisfiability Solutions are truth assignments.
Change one variable from true to false or from false to true.

Resolving the conjecture

Conjecture [Ito et al. 2011]

Not all classical problems in P give rise to polynomially-solvable reconfiguration problems.

Theorem

There exist:

- a classical problem in P that gives rise to a PSPACE-hard reachability problem (shortest paths [Bonsma 2012]), and
- an NP-hard classical problem that gives rise to a polynomially-solvable reachability problem (3-Colouring [Johnson et al. 2014]).

Open question

What properties of problems result in the pattern holding or not?

In search of algorithm paradigms

Contracted solution graph method:

- Shortest path on planar graphs [Bonsma 2012]
- List-colouring on caterpillars [Hatanaka, Ito, Zhou 2014]
- Colouring on $(k-2)$-connected chordal graphs [Bonsma and Paulusma 2015]

Idea:

- Find a way of compacting the information about the reconfiguration graph
- Apply dynamic programming
- Use tree decompositions

Conjecture

Reachability problems are tractable when restricted to graphs of bounded treewidth.

H-word reconfiguration

Given an alphabet and a binary relation between symbols, an **H-word** is a word in which each pair of consecutive symbols is in the relation.

H-word reconfiguration [Wrochna 2014]

Solution: *H*-word

Step: Change one symbol

PSPACE-completeness based on the classic proof of undecidability for general Thue systems. [Post 1947]

Resolving the treewidth conjecture

For graphs of bounded bandwidth (and hence treewidth), the following reconfiguration problems are PSPACE-complete:

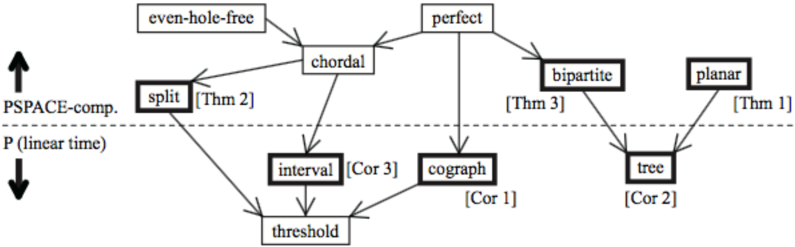
- shortest path
- colouring
- list colouring
- independent set
- vertex cover
- feedback vertex set
- odd cycle transversal
- induced forest
- induced bipartite subgraph
- dominating set

[Wrochna 2014] [Mouawad, Nishimura, Raman, Wrochna 2014] [Haddadan, Ito, Mouawad, Nishimura, Ono, Suzuki, Tebbal 2015]

A graph of bandwidth b has pathwidth and treewidth at most b. [Schiex 1999]

Dominating set intractability/tractability boundary

Key idea: Lots of work to be done on individual problems.
[Haddadan, Ito, Mouawad, Nishimura, Ono, Suzuki, Tebbal 2015]



Open question

What are the boundaries between tractability and intractability for specific problems, and why?

Connectivity case study: Dominating set

$\gamma(G)$ is the *domination number*, the minimum cardinality of a dominating set of G

Gamma graphs [Subramanian and Sridharan 2008]

Solution: Minimum dominating set (γ-set)

Step: Token jumping

Gamma graphs [Fricke et al. 2011]

Solution: Minimum dominating set (γ-set)

Step: Token sliding

***k*-dominating graph [Haas and Seyffarth 2014]**

Solution: Dominating set of size at most k

Step: Token addition and removal

Results on the k-dominating graph

$\Gamma(G)$ is the *upper domination number*, the maximum cardinality of a minimal dominating set of G

The k-dominating graph (a.k.a. reconfiguration graph) is connected when:

- $k \geq \min\{|V(G)| - 1, \Gamma(G) + \gamma(G)\}$ [Haas and Seyffarth 2014]
- bipartite and chordal, when $k \geq \Gamma(G) + 1$ [HS14]
- $k = n - 1$ and there is a matching of cardinality at least two [HS14]
- $k = n - \mu$ and there is a matching of cardinality at least $\mu + 1$ [Suzuki, Mouawad, Nishimura 2014]
- $k = \Gamma(G) + 1$ for certain classes of well-covered graphs [Haas and Seyffarth 17]
- $k = \Gamma(G) + 1$ for graphs that are both perfect and irredundant perfect [HS17]

But, it can be disconnected for $k = \Gamma(G) + 1$, even for planar, bounded tree-width, or b-partite for $b \geq 3$ [SMN14]

Other results on dominating graphs

Key idea: There are many other structural problems worth exploring.

Q: Which graphs are dominating graphs? [Haas and Seyffarth 2014]

A: Only C_6 and C_8 among cycles [Alikhani, Fatehi, Klavžar 2016]

Q: Which graphs have the dominating graph isomorphic to the graph itself? [Haas and Seyffarth 2014]

A: $k = 2$ and G is a star [Alikhani, Fatehi, Klavžar 2016]

There is an infinite family of graphs with exponential diameter for $\gamma(G) + 1$. [Suzuki, Mouawad, Nishimura 2014]

Connectivity case study: Colouring

Vertex colouring reconfiguration

Solution: A k-colouring

Step: Change the colour of one vertex

Mixing

A graph is **k-mixing** if its reconfiguration graph for k colours is connected.

$\Delta(G)$ is the maximum degree of any vertex in G.

- Every graph is $(\Delta(G)+2)$-mixing. [Jerrum 1995]
- Every graph is $(\chi_g(G)+1)$-mixing for $\chi_g(G)$ the Grundy number of G, the highest possible number of colours using by a greedy colouring of G ($\chi_g(G) \leq \Delta(G)+1$). [Bonamy and Bousquet 2013]
- Every graph is $(tw(G)+2)$-mixing for $tw(G)$ the treewidth of G. [Bonamy and Bousquet 2013]

Diameter case study: Colouring

Property	**Diameter bound**	**Result**
$k = 3$	$O(n^2)$ for any component	[Cereceda, van den Heuvel, Johnson 2011]
$k \geq tw(G) + 2$	$2(n^2 + 2)$	[Bonamy and Bousquet 2013]
$k \geq \chi_g(G) + 1$	$4\chi_g(G)n$	[Bonamy and Bousquet 2013]
$k = \Delta + 1$, $\Delta \geq 3$	$O(n^2)$ component (plus isolated vertices)	[Feghali, Johnson, Paulusma 2015]

Various classes of graphs for which reachability is PSPACE-complete have examples with superpolynomial diameter. [Bonsma and Cereceda 2007]

More on colouring

Reconfiguration using **Kempe changes** (exchanging of the colours *a* and *b* in a maximal connected subgraph in which all vertices are coloured either *a* or *b*). [Feghali, Johnson, Paulusma 2015] Used to find clash-free timetables. [Mühlenthaler and Wanka 2015]

Homomorphism reconfiguration is a generalization of colouring. [Wrochna 2014]

Upcoming CanaDAM talk

Benjamin Moore "Some observations on circular colouring mixing for (p,q)-colourings when $p/q < 4$"
Today at 3:50 in this room

Upcoming CanaDAM talk

Karen Seyffarth "Reconfiguring Vertex Colourings of 2-trees"
Today at 4:20 in this room

Working with the diameter

Frequent pattern

Diameter is exponential $\Rightarrow$ reachability PSPACE-hard
Diameter is polynomial $\Rightarrow$ reachability in P

Dealing with diameter

"Neutralize" the effect of diameter by considering the complexity in terms of the length of the reconfiguration sequence.

The **parameterized complexity** of an algorithm is viewed as a function of the input and a parameter.

A problem is **fixed-parameter tractable** (in **FPT**) if the running time is polynomial in the size of the input and a computable function of the parameter. [Downey and Fellows 1997]

Parameterized complexity

Idea: "Push" the non-polynomial part of the complexity onto the parameter(s).

Complexity hierarchy:
FPT $\subseteq$ W[1] $\subseteq$ W[2] $\subseteq \cdots \subseteq$ W[P] $\subseteq$ XP

Classical complexity	**Parameterized complexity**
P	FPT
NP-hard	W[1]-hard (or worse)

Possible parameters:

- Length of reconfiguration sequence (ℓ)
- Bound on a feasible solution (usually size k)
- Properties of the input (e.g. treewidth t)

Relating classical and reconfiguration problems

The **subset problem for hereditary property π** is the problem of finding a subset $V' \subseteq V$ such that $G[V'] \in \pi$.

Hereditary property subset reconfiguration

Solution: A set of vertices U of size at most k such that $G[U]$ has property π

Step: Add or delete a vertex

Hereditary property deletion reconfiguration

Solution: A set of vertices U of size at most k such that $G[V(G) \setminus U]$ has property π

Step: Add or delete a vertex

Example: Independent set as subset, Vertex cover as deletion

General results on parameterized problems

Lemma [Mouawad, Nishimura, Raman, Simjour, Suzuki 2013]

For hereditary property π that satisfies certain conditions, the following are at least as hard as the classical subset problem:

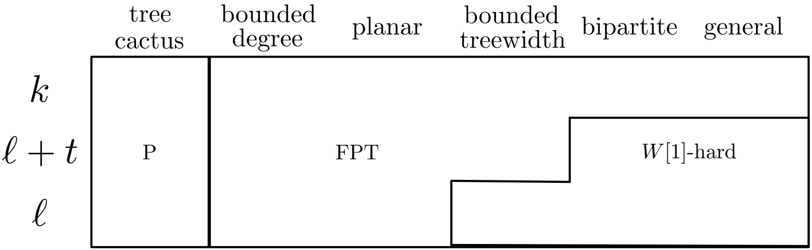
- subset reconfiguration reachability parameterized by $k + \ell$, and
- deletion reconfiguration reachability parameterized by ℓ.

Corollary

The following are both $W[1]$-hard:

- independent set reachability parameterized by $k + \ell$, and
- vertex cover reachability parameterized by ℓ.

Parameterized complexity case study: Vertex Cover



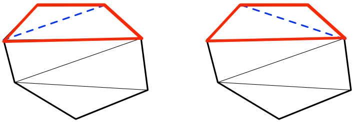
[Ito, Demaine, Harvey, Papadimitriou, Sideri, Uehara, Uno 2011]
[Mouawad, Nishimura, Raman, Simjour, Suzuki 2014]
[Mouawad, Nishimura, Raman 2014]
[Mouawad, Nishimura, Raman, Wrochna 2014]

Shortest transformation case study: Flip distance

Flip distance

Solution: Triangulation of a set of points in the plane

Step: Flip a diagonal of a quadrilateral



Reachability, *connectivity* and *diameter* solved
(always connected, quadratic) [Lawson 1972]
Shortest transformation NP-complete [Lubiw and Pathak 2012]

Diameter and shortest transformation

Polynomial diameter $\not\Rightarrow$ shortest transformation in P

Subsequent work on flip distance

Original problem:

Shortest transformation APX-hard [Pilz 2014], and fixed-parameter tractable with respect to the flip distance [Kanj and Xia 2015]

Extended to labelled flips:

Connectivity and *shortest transformation* results [Lubiw, Masárová, Wagner 2017]

Upcoming CanaDAM talk

Anna Lubiw "Flipping Edge-Labelled Triangulations"
Thursday at 11:20 in ENG 105
Mini-symposium on Topological and Geometric Algorithms, Part II (CM22).

Case study: Satisfiability

Satisfiability reconfiguration

Solution: Satisfying truth assignment

Step: Change one variable from true to false or from false to true

Classical problem Dichotomy: in P for formulas built from **Schaefer relations**, otherwise NP-complete [Schaefer 1978]

Reachability and connectivity Dichotomy: in P for formulas built for **tight relations** (a superset of Schaefer relations), otherwise PSPACE-complete [Gopalan et al. 2009], [Schwerdtfeger 2013]

Diameter Dichotomy: linear if reachability is in P, otherwise exponential

Shortest transformation case study: Satisfiability

- Trichotomy: [Mouawad, Nishimura, Pathak, Raman 2014]
  - P for **navigable** (a subset of tight relations),
  - NP-complete for tight but not navigable, and
  - otherwise PSPACE-complete
- Generalized to “quantum version” [Gharibian and Sikora 2014]
- Generalized to constraint satisfaction [Wrochna 2015]

Other research directions: Other problem domains

Upcoming CanaDAM talk

Moritz Mühlenthaler “Reconfiguration of Common Independent Sets of Partition Matroids”
Today at 4:50 in this room

Past CanaDAM talk

Anna Lubiw “Reconfiguring Ordered Bases of a Matroid”
This morning

Examining adjacency

Problems are **equivalent** if an instance is solvable using one type of reconfiguration step if and only if it is solvable using another type of reconfiguration step.

- Independent Set: TJ and TAR are equivalent [Kamiński et al. 2012]
- Clique: TJ, TS, and TAR are equivalent [Ito et al. 2015]

New definitions of adjacency [deBerg, Jansen, Mukherjee 2016]

- *Multiple Token Jumping* (MTJ)
- *TAR Reconfiguration Threshold*

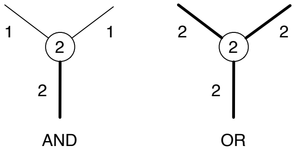
Machinery for PSPACE-completeness results

A *Nondeterministic Constraint Logic (NCL) machine* is an undirected graph with integer weights on vertices and edges. [Hearn and Demaine 2005]

NCL reconfiguration

Solution: An orientation of the edges such that the weight of each vertex is no greater than the sum of the weights of incoming edges

Step: Change the direction of one edge



PSPACE-complete by a reduction from Quantified Boolean Formula [Garey and Johnson 1990]

Toolkit for proving hardness results

Main tools:

- Mimic hierarchy of reductions for classical problems
- Use NCL directly or indirectly
- Use H-word reconfiguration directly or indirectly

NCL remains PSPACE-complete for graphs of bounded bandwidth [van der Zanden 2015]

Generalization [Osawa, Suzuki, Ito, Zhou 2017]:

- Generalized to include neutral, undirected edges
- Introduce new link gadget
- Completes classification of complexity of list edge-colouring, P for $k \leq 3$ and PSPACE-complete otherwise

New approaches to reconfiguration

Optimization What is the optimal solution reachable from a given solution?
Related work on colouring, going from an improper to a proper colouring [Felsner et al. 2009] [Garnero et al. 2016]

Isolated vertices How can the **frozen configurations** be characterized?
Measuring girth What is known about the girth of reconfiguration graphs?

Upcoming CanaDAM talk

Beth Novick "Structural Properties of Shortest Path Graphs"
Today at 5:20 in this room

Open questions

Patterns to explore:

- Relation between classical complexity and reconfiguration complexity (exceptions known)
- Relation between diameter and complexity (flip distance is an exception)
- Relation between symmetric difference and shortest transformation (SAT is an exception)

Open question

What properties of problems result in the pattern holding or not?

Are there classes of graphs for which reconfiguration is generally tractable?
What algorithm paradigms suit reconfiguration problems?

Where to learn more about reconfiguration

- More talks this session and tomorrow
- “The Complexity of Change” [van den Heuvel 2013]
- Report on Banff International Research Station workshop on Combinatorial Reconfiguration, 2017
- Reconfiguration mailing list
  `lists.uwaterloo.ca/mailman/listinfo/reconf`
- Reconfiguration web portal
  `http://www.ecei.tohoku.ac.jp/alg/core/`