CS 360
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Introductory material

1 Strings

substring A string x is a substring of a string y if there exist strings w and z such that $y = wxz$.

prefix A string x is a prefix of a string y if there exists a string z such that $y = xz$.

suffix A string x is a suffix of a string y if there exists a string w such that $y = wx$.

$n_a(x)$ For a a symbol and x a string, $n_a(x)$ is the number of occurrences of a in x.

2 Set theory

You are expected to be familiar with basics of set theory. A few relevant facts are listed here as a reminder. In the following, A and B are subsets of a universal set U. Typically, sets will be languages (hence sets of strings).

Complement The complement of A, denoted $\overline{A}$, is defined as $\overline{A} = \{x \in U \mid x \notin A\}$.

Set difference The difference $A \setminus B$ is defined to be $A \setminus B = A \cap \overline{B}$.

De Morgan laws $\overline{A \cup B} = \overline{A} \cap \overline{B}$; $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Other useful laws Union and intersection are commutative and associative, and each distributes over the other.

3 Operations on languages

concatenation $L_1L_2 = \{xy \mid x \in L_1, y \in L_2\}$

power L^k is k copies of L concatenated; $L^0 = \{\epsilon\}$ and $L^k = LL^{k-1}$.

Kleene star $L^* = \cup_{k=0}^{\infty} L^k$; also $L^+ = \cup_{k=1}^{\infty} L^k$.

4 Closure properties

A set S is closed under an operation if the element derived by applying the operation to any element(s) of S yields an element of S. That is, for a unary operation $\diamond$, $\diamond(x)$ is in S for any $x \in S$, and for a binary operation $\diamond$, $x \diamond y$ is in S for any $x, y \in S$.

The textbook uses closure in this sense as well as to specifically mean closure under the operation Kleene star. To avoid confusion, I will only use closure in this more general sense.

KEY POINT (and common pitfall): It is important to observe that the operation will be an operation on elements of S. If S is a language, we can talk about the closure of S under an operation on strings. If S is a class of languages (hence of set of languages), then we can talk about the closure of S under an operation on languages.

5 Induction

You may find it useful to use a bit more structure in your proofs by induction by stating explicitly both your assumptions and the statement to prove. The example below is taken from lecture, and demonstrates one way of presenting a proof by induction.

We prove by induction that if $L^2 \subseteq L$, then $L^+ \subseteq L$. First, we transform the statement to prove into one in which the "n" is more obvious: "For every $n \geq 1$, if $L^2 \subseteq L$, then $L^n \subseteq L$". Our proof is by induction on the size of n.

Basis. For $n = 1$, we prove that if $L^2 \subseteq L$, then $L^1 \subseteq L$. The result follows from the facts that $L^1 = L$ and that for any set S, $S \subseteq S$. Notice that we did not use the fact that $L^2 \subseteq L$ anywhere in our proof, as it was not needed.

Induction hypothesis. For $n = k$, if $L^2 \subseteq L$, then $L^n \subseteq L$.

Statement to prove. For $n = k + 1$, if $L^2 \subseteq L$, then $L^n \subseteq L$.

Inductive step. As in the statement to prove, we let $n = k + 1$ and assume that $L^2 \subseteq L$.

Applying the induction hypothesis, since $L^2 \subseteq L$, we can conclude that $L^k \subseteq L$. Moreover, for any sets A, B, and C, $A \subseteq B$ implies $CA \subseteq CB$. By setting $A = L^k$, $B = L$, and $C = L$, we can conclude that $LL^k \subseteq LL = L^2$.

We observe by the recursive definition of $L^n = L^{k+1}$ that $L^{k+1} = LL^k$. Thus, $LL^k \subseteq L^2$ implies that $L^{k+1} \subseteq L^2$.

Finally, using again the assumption that $L^2 \subseteq L$, we combine $L^{k+1} \subseteq L^2$ and $L^2 \subseteq L$ to conclude that $L^{k+1} \subseteq L$, as claimed.