CS 886: Multiagent Systems

Computing Equilibria

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Outline

Dominant and Dominated Strategies

For the time being, let us restrict ourselves to pure strategies.

Definition

Strategy s_i is a strictly dominant strategy if for all $s_i' \neq s_i$ and for all s_{-i}

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$$

Prisoner's Dilemma

	C	D
C	-1,-1	-4,0
D	0, -4	-3,-3

Dominant-strategy equilibria

Dominated Strategies

Definition

A strategy s_i is strictly dominated if there exists another strategy s_i' such that for all s_{-i}

$$u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$$

Definition

A strategy s_i is weakly dominated if there exists another strategy s_i' such that for all s_{-i}

$$u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i})$$

with strict inequality for some s_{-i}.

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Example

	L	R
U	1,-1	-1,1
M	-1,1	1,-1
D	-2,5	-3,2

D is strictly dominated

	L	R
U	5,1	4,0
M	6,0	3,1
D	6,4	4,4

U and M are weakly dominated

Iterated Deletion of Strictly Dominated Strategies

Algorithm

- Let R_i be the removed set of strategies for agent i
- $R_i = \emptyset$
- Loop
  - Choose i and s_i such that $s_i \in A_i \setminus R_i$ and there exists s_i' such that
  $$u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}) \ \forall s_{-i}$$
  - Add s_i to R_i
  - Continue

Example

	R	C	L
U	3,-3	7,-7	15, -15
D	9,-9	8,-8	10,-10

Some Results

Theorem

If a unique strategy profile s^* survives iterated deletion then it is a Nash equilibrium.

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Weakly dominated strategies cause some problems.

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Domination and Mixed Strategies

The definitions of domination (both strict and weak) can be easily extended to mixed strategies in the obvious way.

Theorem

Agent i's pure strategy s_i is strictly dominated if and only if there exists another (mixed) strategy σ_i such that

$$u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i})$$

for all s_{-i}.

Example

	L	R
U	10,1	0,4
M	4,2	4,3
D	0,5	10,2

Strategy $(\frac{1}{2}, 0, \frac{1}{2})$ strictly dominates pure strategy M.

Theorem

If pure strategy s_i is strictly dominated, then so is any (mixed) strategy that plays s_i with positive probability.

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Theorem

If pure strategy s_i is strictly dominated, then so is any (mixed) strategy that plays s_i with positive probability.

Maxmin and Minmax Strategies

- A **maxmin strategy** of player i is one that maximizes its worst case payoff in the situation where the other agent is playing to cause it the greatest harm

$$\arg\max_{s_i}\min_{s_{-i}} u_i(s_i, s_{-i})$$

- A **minmax strategy** is the one that minimizes the maximum payoff the other player can get

$$\arg\min_{s_i}\max s_{-i} u_{-i}(s_i, s_{-i})$$

Example

In 2-player games, maxmin value of one player is equal to the minmax value of the other player.

	L	R
U	2,3	5,4
D	0,1	1,2

Calculate maxmin and minmax values for each player (you can restrict to pure strategies).

Zero-Sum Games

- The maxmin value of one player is equal to the minmax value of the other player
- For both players, the set of maxmin strategies coincides with the set of minmax strategies
- Any maxmin outcome is a Nash equilibrium. These are the only Nash equilibrium.

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Solving Zero-Sum Games

Let U_i^* be unique expected utility for player i in equilibrium. Recall that $U_1^* = -U_2^*$.

$$\begin{array}{lll} \text{minimize} & U_1^* & \\ \text{subject to} & \sum_{a_k \in A_2} u_1(a_j, a_k) s_2(a_k) \leq U_1^* & \forall a_j \in A_1 \\ & \sum_{a_k \in A_2} s_2(a_k) = 1 & \\ & s_2(a_k) \geq 0 & \forall a_k \in A_2 \end{array}$$

LP for 2's mixed strategy in equilibrium.

Solving Zero-Sum Games

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Recall that $U_1^* = -U_2^*$.

$$\begin{array}{lll} \text{maximize} & U_1^* & \\ \text{subject to} & \sum_{a_j \in A_1} u_1(a_j, a_k) s_1(a_j) \geq U_1^* & \forall a_k \in A_2 \\ & \sum_{a_j \in A_1} s_1(a_j) = 1 & \\ & s_1(a_j) \geq 0 & \forall a_j \in A_1 \end{array}$$

LP for 1's mixed strategy in equilibrium.

Two-Player General-Sum Games

LP formulation does not work for general-sum games since agents' interests are no longer diametrically opposed.

Linear Complementarity Problem (LCP)
Find any solution that satisfies

$$
\begin{array}{ll}
\sum_{a_k \in A_2} u_1(a_j, a_k) s_2(a_k) + r_1(a_j) = U_1^* & \forall a_j \in A_1 \\
\sum_{a_j \in A_1} u_2(a_j, a_k) s_1(a_j) + r_2(a_k) = U_2^* & \forall a_k \in A_2 \\
\sum_{a_j \in A_1} s_1(a_j) = 1 \ \sum_{a_k \in A_2} s_2(a_k) = 1 & \\
s_1(a_j) \geq 0, s_2(a_k) \geq 0 & \forall a_j \in A_1, a_k \in A_2 \\
r_1(a_j) \geq 0, r_2(a_k) \geq 0 & \forall a_j \in A_1, a_k \in A_2 \\
r_1(a_j) s_1(a_j) = 0, r_2(a_k) s_2(a_k) = 0 & \forall a_j \in A_1, a_k \in A_2
\end{array}
$$