CS 886: Multiagent Systems Computing Equilibria

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Outline



Dominant and Dominated Strategies





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Dominant and Dominated Strategies

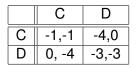
For the time being, let us restrict ourselves to pure strategies.

Definition

Strategy s_i is a strictly dominant strategy if for all $s'_i \neq s_i$ and for all s_{-i}

$$u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$$

Prisoner's Dilemma



Dominant-strategy equilibria

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Dominated Strategies

Definition

A strategy s_i is strictly dominated if there exists another strategy s'_i such that for all s_{-i}

$u_i(s'_i, s_{-i}) > u_i(s_i, s_{-i})$

Definition

A strategy s_i is weakly dominated if there exists another strategy s'_i such that for all s_{-i}

 $u_i(s'_i, s_{-i}) \geq u_i(s_i, s_{-i})$

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with strict inequality for some s_{-i} .

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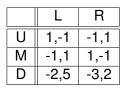
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with strict inequality for some s_{-i} .

Example



	L	R
U	5,1	4,0
М	6,0	3,1
D	6,4	4,4

D is strictly dominated

U and M are weakly dominated

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Iterated Deletion of Strictly Dominated Strategies

Algorithm

- Let R_i be the removed set of strategies for agent i
- $R_i = \emptyset$
- Loop
 - Choose *i* and *s_i* such that *s_i* ∈ *A_i* \ *R_i* and there exists *s'_i* such that

 $u_i(s'_i, s_{-i}) > u_i(s_i, s_{-i}) \ \forall s_{-i}$

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- Add s_i to R_i
- Continue

Example

	R	С	L
U	3,-3	7,-7	15, -15
D	9,-9	8,-8	10,-10

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Some Results

Theorem

If a unique strategy profile s^{*} survives iterated deletion then it is a Nash equilibrium.

Theorem

If *s*^{*} is a Nash equilibrium then it survives iterated elimination.

Weakly dominated strategies cause some problems.

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Domination and Mixed Strategies

The definitions of domination (both strict and weak) can be easily extended to mixed strategies in the obvious way.

Theorem

Agent i's pure strategy s_i is strictly dominated if and only if there exists another (mixed) strategy σ_i such that

 $u_i(\sigma_i, \mathbf{s}_{-i}) > u_i(\mathbf{s}_i, \mathbf{s}_{-i})$

for all s_{-i} .

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Example

	L	R
U	10,1	0,4
М	4,2	4,3
D	0,5	10,2

Strategy $(\frac{1}{2}, 0, \frac{1}{2})$ strictly dominates pure strategy *M*.

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Theorem

If pure strategy s_i is strictly dominated, then so is any (mixed) strategy that plays s_i with positive probability.

Example

	L	R
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Strategy $(\frac{1}{2}, 0, \frac{1}{2})$ strictly dominates pure strategy *M*.

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Theorem

If pure strategy s_i is strictly dominated, then so is any (mixed) strategy that plays s_i with positive probability.

Maxmin and Minmax Strategies

• A **maxmin strategy** of player *i* is one that maximizes its worst case payoff in the situation where the other agent is playing to cause it the greatest harm

 $\arg\max_{s_i}\min_{s_{-i}}u_i(s_i,s_{-i})$

• A **minmax strategy** is the one that minimizes the maximum payoff the other player can get

 $\arg\min_{s_i}\max s_{-i}u_{-i}(s_i,s_{-i})$

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Example

In 2-player games, maxmin value of one player is equal to the minmax value of the other player.

	L	R
U	2,3	5,4
D	0,1	1,2

Calculate maxmin and minmax values for each player (you can restrict to pure strategies).

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Zero-Sum Games

• The maxmin value of one player is equal to the minmax value of the other player

- For both players, the set of maxmin strategies coincides with the set of minmax strategies
- Any maxmin outcome is a Nash equilibrium. These are the only Nash equilibrium.

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Solving Zero-Sum Games

Let U_i^* be unique expected utility for player *i* in equilibrium. Recall that $U_1^* = -U_2^*$.

$$\begin{array}{ll} \text{minimize} & U_1^* \\ \text{subject to} & \sum_{a_k \in A_2} u_1(a_j, a_k) s_2(a_k) \leq U_1^* \quad \forall a_j \in A_1 \\ & \sum_{a_k \in A_2} s_2(a_k) = 1 \\ & s_2(a_k) \geq 0 \qquad \qquad \forall a_k \in A_2 \end{array}$$

LP for 2's mixed strategy in equilibrium.

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Solving Zero-Sum Games

Let U_i^* be unique expected utility for player *i* in equilibrium. Recall that $U_1^* = -U_2^*$.

$$\begin{array}{ll} \text{maximize} & U_1^* \\ \text{subject to} & \sum_{a_j \in A_1} u_1(a_j, a_k) s_1(a_j) \geq U_1^* \quad \forall a_k \in A_2 \\ & \sum_{a_j \in A_1} s_1(a_j) = 1 \\ & s_1(a_j) \geq 0 \qquad \qquad \forall a_j \in A_1 \end{array}$$

LP for 1's mixed strategy in equilibrium.

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Two-Player General-Sum Games

LP formulation does not work for general-sum games since agents' interests are no longer diametrically opposed.

Linear Complementarity Problem (LCP) Find any solution that satisfies

$$\begin{array}{ll} \sum_{a_k \in A_2} u_1(a_j, a_k) s_2(a_k) + r_1(a_j) = U_1^* & \forall a_j \in A_1 \\ \sum_{a_j \in A_1} u_2(a_j, a_k) s_1(a_j) + r_2(a_k) = U_2^* & \forall a_k \in A_2 \\ \sum_{a_j \in A_1} s_1(a_j) = 1 & \sum_{a_k \in A_2} s_2(a_k) = 1 \\ s_1(a_j) \ge 0, s_2(a_k) \ge 0 & \forall a_j \in A_1, a_k \in A_2 \\ r_1(a_j) \ge 0, r_2(a_k) \ge 0 & \forall a_j \in A_1, a_k \in A_2 \\ r_1(a_j) s_1(a_j) = 0, r_2(a_k) s_2(a_k) = 0 & \forall a_j \in A_1, a_k \in A_2 \end{array}$$

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