Mathematical Typography

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Outline

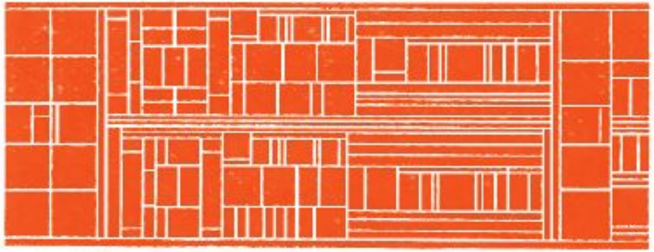
1. History of mathematical typography
2. Mathematics for TeX
3. Mathematics for Metafont
4. Math font requirements
5. Diversity of math fonts

Section 1:
History of Mathematical Typography

Mathematical typography before the 1900s

- Mathematics was typeset by hand using sorts.
- Typesetters often did not know the subject.
- A logistical problem of needing bold, italic, sans-serif sorts at the same time.
- Two dimensional arrangement of sorts was a considerable mechanical challenge.

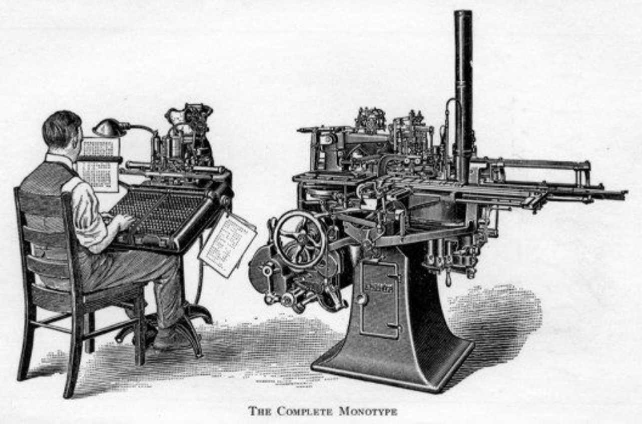
$$\breve{x} = \frac{\int_0^a x dx \int_0^{\sqrt{a^2 - x^2}} r\sqrt{x^2 + r^2} dr}{\int_0^a dx \int_0^{\sqrt{a^2 - x^2}} r\sqrt{x^2 + r^2} dr} = \frac{2a}{5}$$



Mathematical typography till the 1950s

- Hot-metal composition using Monotype machines.
- The first mechanisation of mathematical typesetting.
- The results were exceptionally good.
- But typesetting was extremely complicated: required multiple passes to type an equation.

$$\prod_{1 \le i \le n} \left(\sum_{1 \le j \le n} a_{ij}^{\;2} \right)^{1/2}$$



https://letterpresscommons.com/monotype/

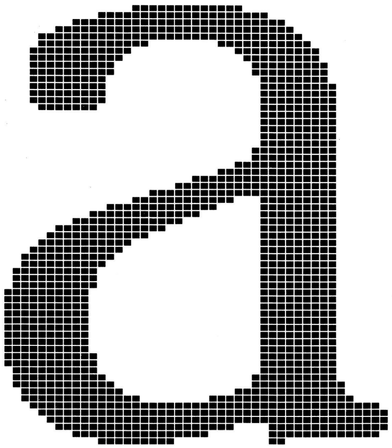
Phototypesetting during the 1960s

- Typesetting on Monotype was costly, and skilled compositors were retiring.
- Photo-typesetting started to appear in the 1960s.
- But phototypesetters were limited in the number of sorts, and was not suitable for complex mathematics.
- The quality of output was very poor compared to Monotype.

http://graficnotes.blogspot.com/2017/01/phototypesetting.html

Digital typography in the 1970s

- A radical change in printing technology occurred from analog to digital.
- Digital printers could produce high-quality print because dots of ink bleed to form smooth looking curves.
- The problem of printing changed from metallurgy to optics to computer science.
- It was at this time that Don Knuth seized the opportunity to develop a digital typesetting system: TeX



https://medium.com/@fpeulrich/a-brief-overview-of-developments-in-digital-type-design-561d9e63a122

Section 2:
Mathematics for TeX

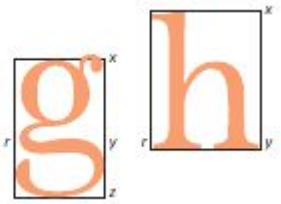
TeX

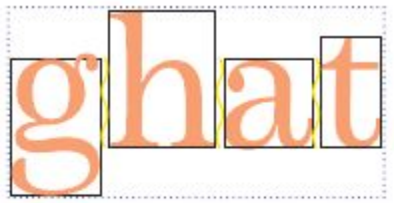
- Knuth developed TeX as a type-setting system capable of doing mathematical work.
- The job of TeX is to put letters and symbols into the right positions on a page.
- TeX allows data-entry in a mathematically-logical order when building equations.
- The author does the type-setting. So if anything goes wrong in printing, the authors have only themselves to blame.

https://en.wikipedia.org/wiki/Donald_Knuth

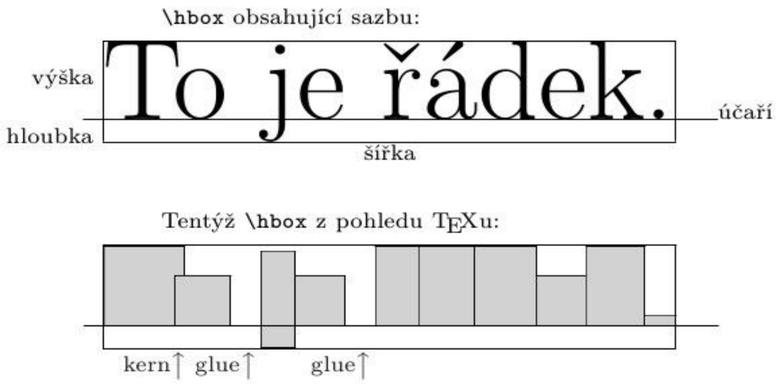
Math for TeX: boxes and glue

- The typographic elements of TeX are *boxes* and *glue*.
- Each character is a box. A line is a box of character boxes. A page is a box of line boxes.
- Character boxes are glued to form word boxes. Word boxes are glued to form line boxes. Line boxes are glued to form paragraph boxes.





The “elastic” glue



- A glue has three components *(x, y, z)* called *space component*, *stretch component* and *shrink component* respectively.

- *x* represents the ideal space between boxes; *y* is the extra space that can be added; *z* is the space that can be removed.

- The glue between letter of a word have x = y = z = 0. The glue between words may have x = y and z = x/2 where x = width of an \`e’.

- Glues with infinite stretch components are used for left/right/center justification.

Line division in TeX

- TeX’s line-breaking algorithm uses the concept of “badness” of glue.
- The badness rating increase with increase in the stretch or shrink values of glues between adjacent boxes.
- There are penalty points on badness for hyphenation, orphan and widow lines, breaking up a mathematical formula, etc.
- The line breaking algorithm must find break points that minimize the sum of squares of badness of each line.

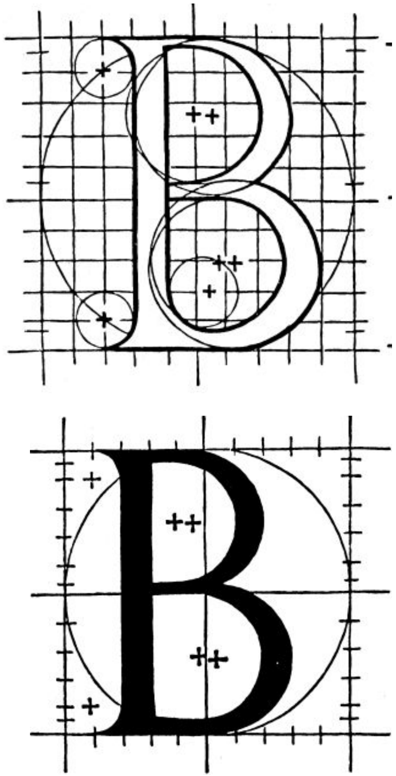
The Knuth-Plass algorithm

- If there are *n* break points in a paragraph, there are 2^n ways to divide the paragraph.
- A dynamic programming algorithm can reduce the number of computational steps to n^2
- For practical purposes, the run time is mostly linear since unlikely break-points are never tested.

Section 3:
Mathematics for Metafont

Mathematical type design

- Mathematical definitions of letter forms have a long history.
- One of the prominent ideas during the 16th century was ruler-and-compass construction of capital letters.
- However, there is a near-universal agreement among scholars of type design that those efforts were a failure.
- Over 400 years since the 16th century, mathematics developed new tricks to draw curves besides ruler-and-compass.



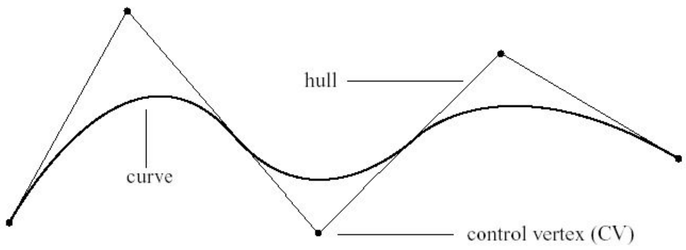
How to draw “pleasing” curves?

- Given *n* points in the plane, what properties must a pleasing closed curve passing through those points have?
- **Invariance:** If the points are translated/rotated/expanded, the curve is also translated/rotated/expanded in the same way.
- **Symmetry:** Cyclic permutation of the points does not change the solution.
- **Extensionality:** Adding a point on the curve does not change the solution.
- **Locality:** The segment of a curve between two points depends only on those two points and the two neighbouring points.

How to draw “pleasing” curves?

- **Smoothness:** The curve has no sharp corners.

- **Roundness:** If 4 consecutive points are on a circle, the most pleasing curve through them is a circle.



- The Metafont system (the system that draws fonts for TeX) uses the concept of cubic splines and control points to compute the curves.

- The algorithms have improved dramatically from $O(n^3)$ to $O(n)$, where n is the number of pixels per inch.

Metafont

0123456789

(Numerals drawn using Metafont)

- Metafont is a schematic description of how to draw a family of fonts.
- The description gives rules in terms of parameters. A different setting of parameters gives a different font. Hence the term meta-font.
- Metafont describes the motion of the center of a “pen” instead of describing the boundary of each character.
- The three P’s of Metafont: *Pens* and *Parameters* via *Programs*!

Metafont: Pens and erasers

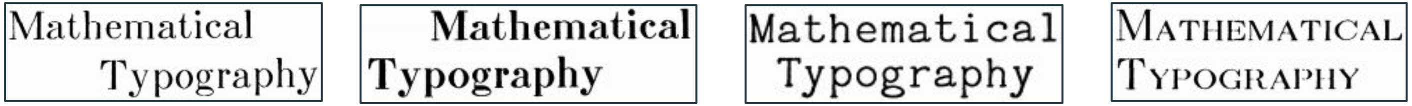
- Pens have an elliptical nib with horizontal axes. The pen’s shape and size are allowed to vary as the pen moves.

- For the most part, only three types of pens are used: *horizontal pen*, *circular pen* and a *vertical pen*.

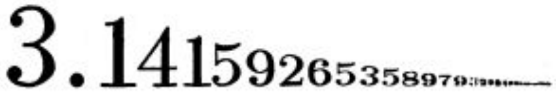
- Metafont also uses erasers to erase strokes drawn by a pen.

Metafont: Parameters

- Metafont has about 20 parameters specifying pen-width, heights of ascenders/descenders, x-width, length of serifs, M-width, etc.

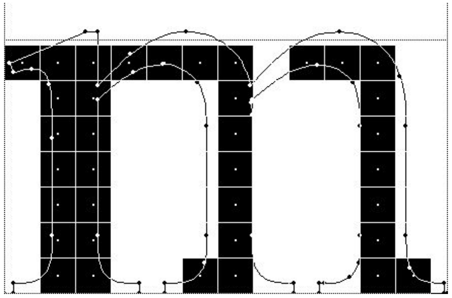


- The parameters of Metafont help in carrying out visual experiments to obtain good parameter settings for each size of type.



Digitising the fonts

- Metafont must also express characters on a discrete raster.
- The technique of rounding pixel values to 0 or 1 produces unsatisfactory results.
- Many such issues are avoided in Metafont because the pen itself is first digitised.
- Digitising the pen is not easy either. For example, how do we “center” a circular 2x2 pen on a grid?



Digitising the pen

- *Rounding rule*: If an elliptical pen of integer width *w* and integer height *h* is to positioned at real coordinates *(x,y)*, its position on the discrete raster is $(\lfloor x - \delta(w) \rfloor, \lfloor y - \delta(h) \rfloor)$, where δ(even) = ½ and δ(odd) = 0.

- The pen consists of all integer pairs *(x,y)* that satisfy

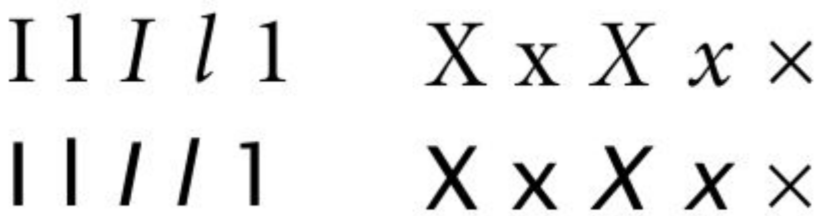
$$\left(\frac{2(x-\delta(w))}{2}\right)^2 + \left(\frac{2(y-\delta(h))}{2}\right)^2 \leqslant 1 + \max\left(\frac{2\delta(w)}{w}, \frac{2\delta(h)}{h}\right)^2.$$

- Later, substantially better rounding rules for pens with interesting connections to number theory and geometry were developed.

Section 4:
Math font requirements

Fonts and optical sizes

- Sans-Serif fonts are less suitable because many letters and symbols are too similar



- Needs at least three different optical sizes.

$A^{A^{A}}$ $i_{i_{i}}$ $x^{x_{x}}$ $(a+b)^{(a+b)^{(a+b)}}$

Math-italic and Greek letters

- For better differentiation from text-italic, math-italic should be wider.

adfhiknvxy

Cambria Italic glyphs

$adfhiknvxy$

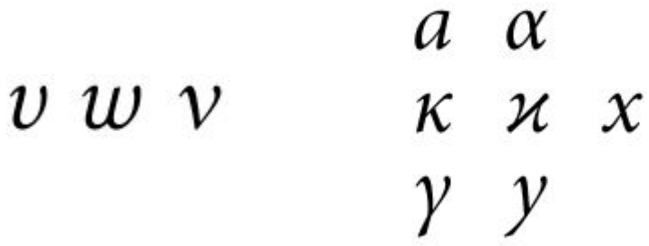
Cambria Math math-italic glyphs

- Mathematics needs a complete set of greek letters in Roman and italic form.

$$\mathrm{\Gamma}\ \mathrm{\Delta}\ \mathrm{\Theta}\ \ldots\ \mathrm{\alpha}\ \mathrm{\beta}\ \mathrm{\gamma}\ \ldots$$
$$\mathit{\Gamma}\ \mathit{\Delta}\ \mathit{\Theta}\ \ldots\ \alpha\ \beta\ \gamma\ \ldots$$

Greek letters and derived glyphs

- Some Greek letterforms need special attention.

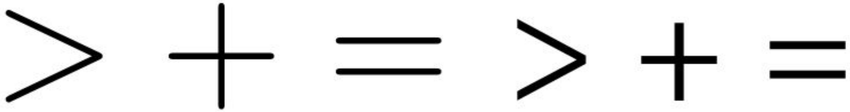


- Some additional derived glyphs are needed.

Centering on axis and pen-shape

- Mathematical symbols must be centered on the axis, except for derived symbols.

- Ends of lines could be rounded or cut-off.



Fractions and side-bearings

- Numerators (resp. denominators) should be high (resp. low) enough to allow for descenders (resp. ascenders) or indices (resp. exponents).

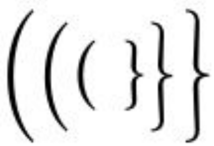
$$\frac{n}{3}\cdot\frac{2}{3}\cdot\frac{a}{3}\cdot\frac{af}{3}\cdot\frac{af_2}{3} \qquad \frac{2}{n}\cdot\frac{2}{3}\cdot\frac{2}{a}\cdot\frac{2}{af}\cdot\frac{2}{a^2}\cdot\frac{2}{af^2}$$

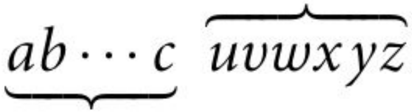
- Side-bearings of characters should be wide-enough so that they are centered and non-touching when enclosed by delimiters.

$$(\; [\; \{ \; \lfloor \; \lceil \; \langle \; / \; | \; \| \; \backslash \; \rangle \; \rceil \; \rfloor \; \} \;] \;) \qquad |d|e|f|g|h|i|j|k|l|m|n|$$

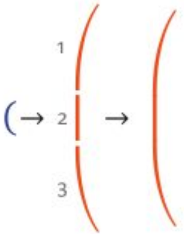
Variants and assemblies

- Delimiters must contain horizontal and vertical variants of different sizes.





- If the largest of the variants is not enough, an assembly needs to be created.

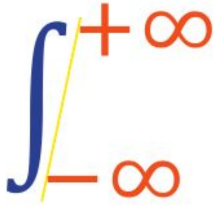


Math kerning and italics correction

- Kerning improves spacing between glyphs, especially between base and script glyphs.

$$\Gamma_2\ \Gamma^2\ \Gamma_2^2 \qquad \Delta_2\ \Delta^2\ \Delta_2^2 \qquad P_2\ P^2\ P_2^2 \qquad f_2\ f^2\ f_2^2$$

- Italics correction is done when an italics glyph appears too close to the following upright glyph or a superscript glyph.



Dotless forms and accent attachment

- Dotless variants of the letters *i* and *j* are needed when an accent needs to be placed above them.

- The mathematical center and the optical center of glyphs seldom match exactly. Hence, accents need to be kerned slightly off-center.

Section 5:
Diversity of math fonts

Math fonts in TeX

- Some examples are Computer Modern, MathTime, Lucida New Math, Mathematical Pi, Euler fonts.

- Each of these fonts contain a different set of glyphs, and the TeX support layer for each of them had to be written from scratch.

- This explains the general dearth of math fonts in TeX.

AMS Euler

$$\int_0^3 9x^2 + 2x + 4\,dx = 3x^3 + x^2 + 4x + C\Big]_0^3 = 102$$

$$e^{x+iy} = e^x(\cos y + i\sin y)$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Unicode maths and OpenType

- The Unicode contains a formal description of thousands of glyphs that a maths font should contain.
- Apart from glyphs, mathematical typesetting needs parameters like spacing, kerning, accent positioning, etc. to place the glyphs in the right places.
- Microsoft's OpenType specification uses Unicode maths glyphs and contains parameters for typesetting in the form of structured tables.
- The OpenType specification generalises and extends the original algorithms of TeX.

The unicode-math package in LaTeX

- The unicode-math package uses Unicode math glyphs and Microsoft's OpenType font format table.
- This package completely replaces LaTeX's math setup. It makes font switching easy just by using **\setmathfont{}** macro.
- It also contains commands for switching between different math styles.

\mathit	abc	XYZ	$\alpha\xi\theta$	$\Psi\Xi\Omega$	
\mathbfit	abc	XYZ	$\alpha\xi\theta$	$\Psi\Xi\Omega$	
\mathup	abc	XYZ	$\alpha\xi\theta$	$\Psi\Xi\Omega$	123
\mathbfup	abc	XYZ	$\alpha\xi\theta$	$\Psi\Xi\Omega$	123
\mathbb	abc	XYZ			123
\mathtt	abc	XYZ			123
\mathsfit	abc	XYZ			
\mathbfsfit	abc	XYZ	$\alpha\xi\theta$	$\Psi\Xi\Omega$	
\mathsf	abc	XYZ			123
\mathbfsfup	abc	XYZ	$\alpha\xi\theta$	$\Psi\Xi\Omega$	123

Challenges with Unicode maths

- Using the correct characters

U+005C	Reverse solidus	x\y	\backslash
U+2216	Set minus	x ∖ y	\smallsetminus
U+29F5	Reverse solidus operator	x ⧵ y	\setminus
U+29F9	Big reverse solidus	x ⧹ y	\xbsol

U+25B5	\vartriangle	x ▵ y	relation
U+25B3	\bigtriangleup	x △ y	binary
U+25B3	\triangle	x△y	ordinary
U+2206	\increment	x∆y	ordinary
U+0394	\mathup\Delta	xΔy	ordinary

- MathML and LaTeX use different names for the same Unicode symbol. For example, ∞ is **\infty** in TeX and **∞** in MathML.

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