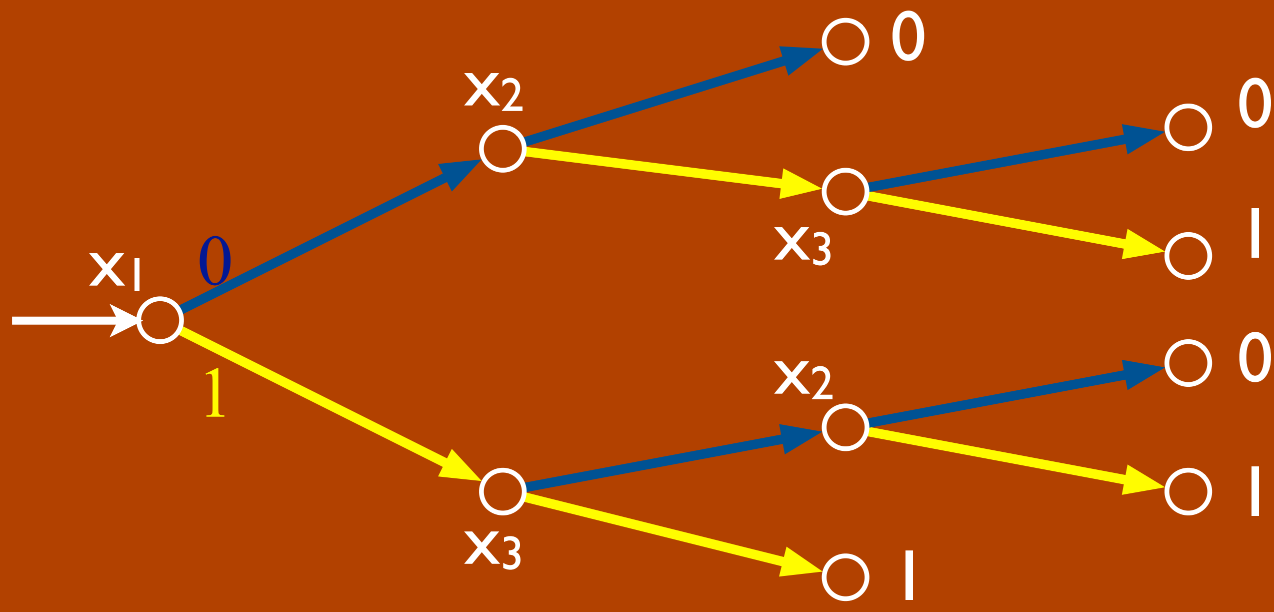
Quantum query complexity

Ben Reichardt

IQC, University of Waterloo

$$f : \mathcal{D} \to E$$

$$\mathcal{D} \subseteq \{0, 1\}^n$$



x_1
0
1
x_2
x_3
0
x_3
x_2
1
0
1
0
1
x=110

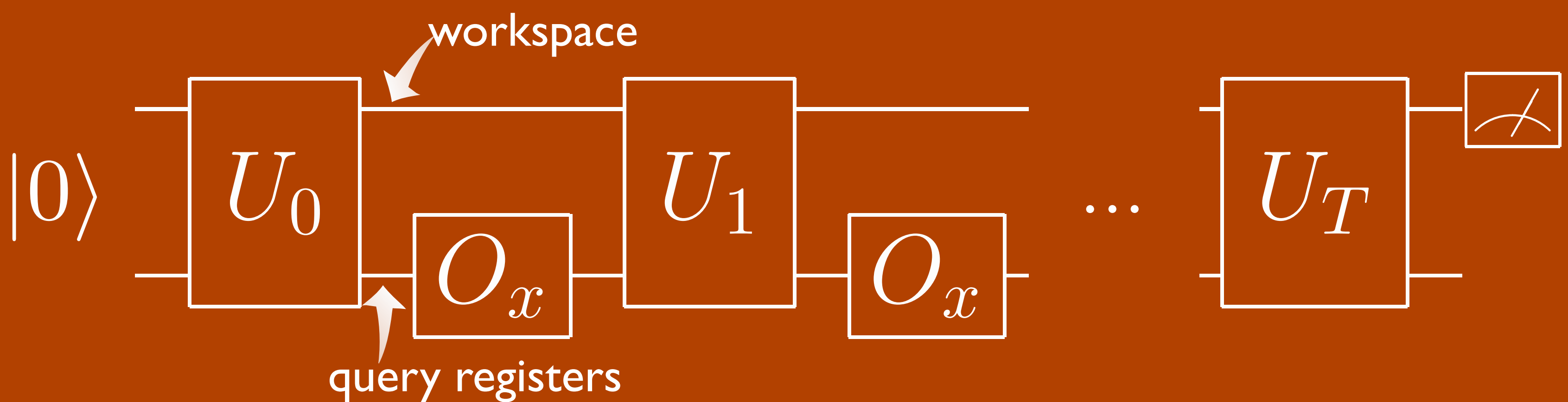
Models:
- Deterministic query/decision-tree complexity
- Randomized query complexity
  - bounded-error/Monte Carlo (R_2), Zero-error/Las Vegas (R_0), one-sided error (R_1)
- Certificate complexity
  - A.K.A. Nondeterministic query complexity
- Quantum query complexity

For total functions (i.e., domain $D=\{0,1\}^n$), all are equivalent up to polynomials, e.g.,

$$D(f) \leq \min\{C^2, R_0^2, R_2^3, R_1 R_2, Q_2^6, Q_{\mathrm{E}}, Q_1 Q_2^2\}$$

see [BBCMW 9802049]

also [AA 0911.0996]



$$O_x|j\rangle = (-1)^{x_j}|j\rangle \quad \text{(with } x_0 = 0\text{)}$$

A. Query complexity

B. Adversary lower bounds

$\boxed{\text{Break}}$

C. Spectra of reflections

D. Adversary upper bound

$$Q(f) = \Theta(\mathrm{Adv}^{\pm}(f))$$

Examples

$$\mathrm{OR} : \{0,1\}^n \rightarrow \{0,1\}$$

$$\mathrm{OR}(x) = \begin{cases} 1 & \text{if } |x| := \sum_{j=1}^{n} x_j \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

OR $\sqrt{n}$ [G 9605043, BBBV 9701001]

$A: \{0,1\}^* \rightarrow \{0,1\}^*$
uniformly random length-preserving function

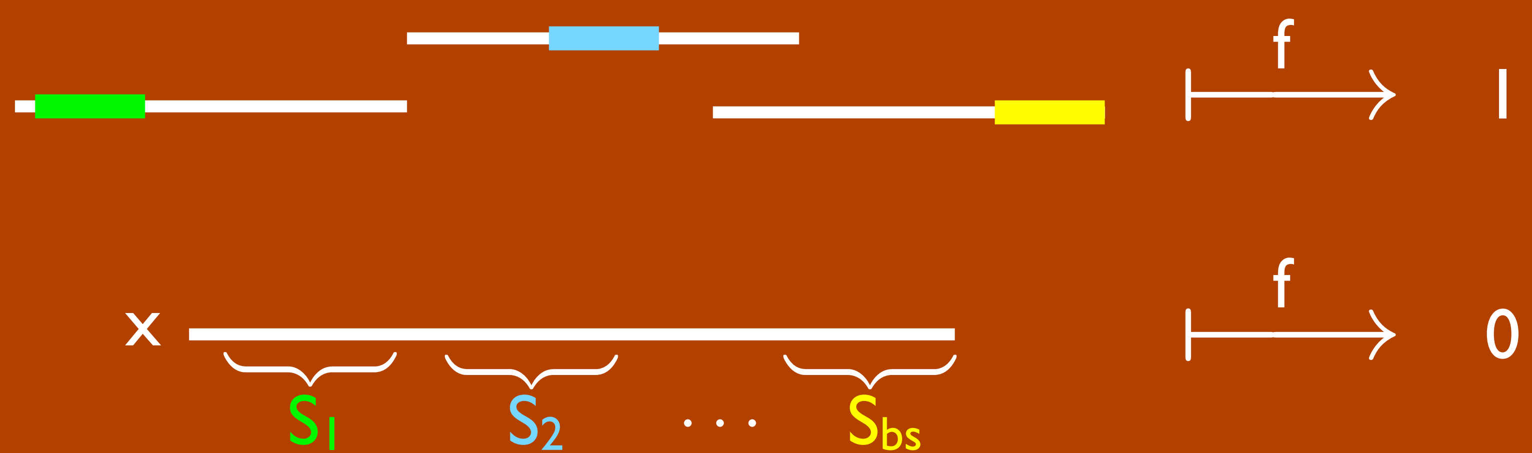
$L_A = \text{Range}(A)$

- $L_A \in NP^A$
- With probability one, $L_A \notin \text{BQTime}(o(\sqrt{2^n}))^A$

OR $\sqrt{n}$ [G 9605043, BBBV 9701001]

Theorem: For $f: \{0,1\}^n \rightarrow \{0,1\}$, $D(f)=O(Q(f)^6)$

Proof: Block sensitivity of x = # of disjoint sets each of which can be flipped to flip f



Just like OR_{bs}! $\Rightarrow \sqrt{bs} \leq Q(f|_{\{x \text{ and its neighbors}\}}) \leq Q(f)$

Rest of proof is classical: $D(f) \leq bs(f)C(f)$, $C(f) \leq bs(f)^2$

OR | $\sqrt{n}$ | [G 9605043, BBBV 9701001]

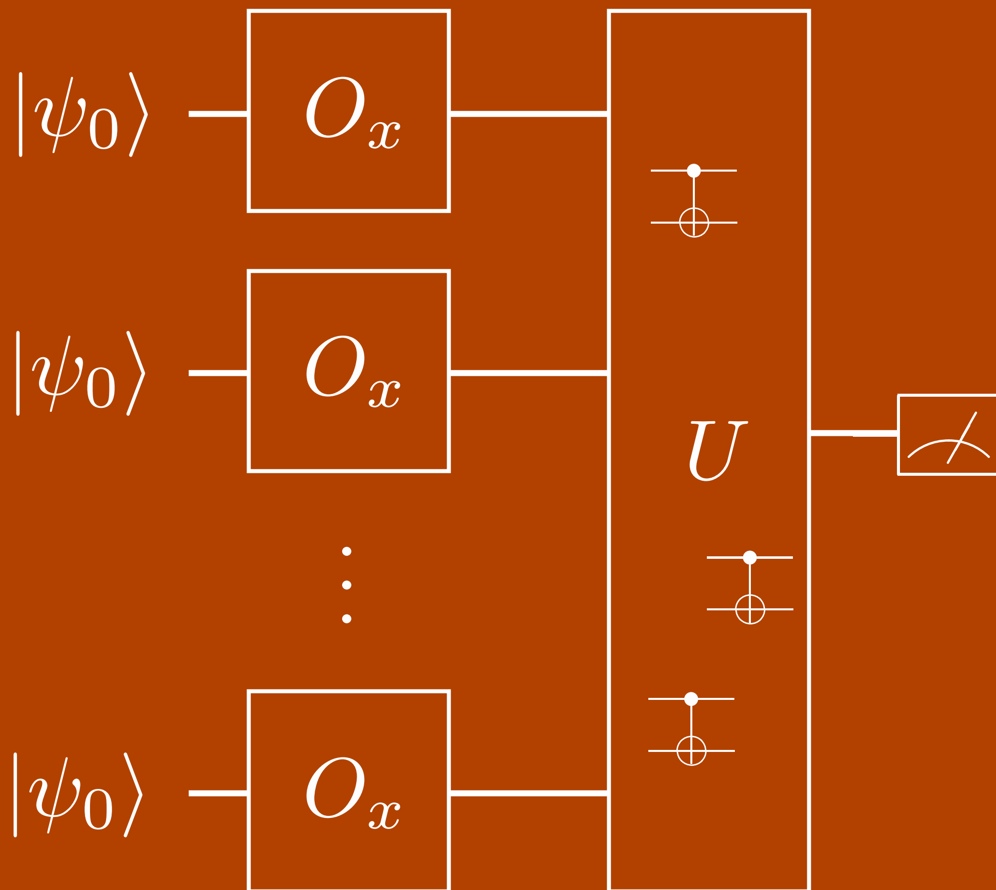
Hidden subgroup | log |G| | [EHK 9901034, KNP 0501060]

Input: $x =$  $\in \Sigma^G$

$$x_g = x_h \Leftrightarrow gh^{-1} \in H$$

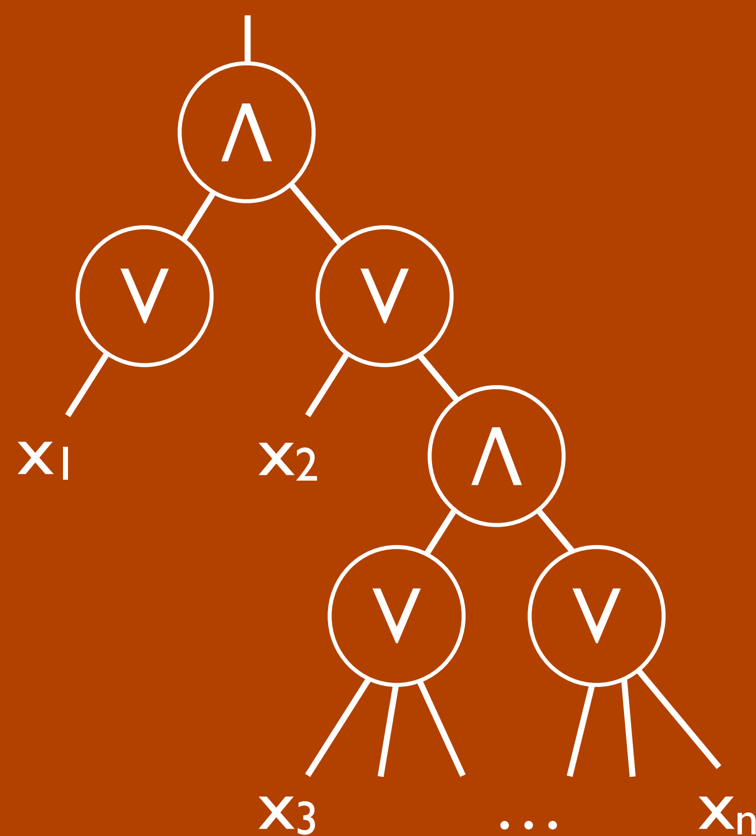
Output: H

OR	$\sqrt{n}$	[G 9605043, BBBV 9701001]
Hidden subgroup	$\log \lvert G \rvert$	[EHK 9901034, KNP 0501060]



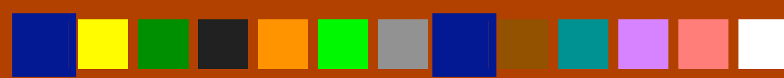
Time complexity > Query complexity (can be $\gg$!)

OR	$\sqrt{n}$	[G 9605043, BBBV 9701001]
Hidden subgroup	$\log \lvert G \rvert$	[EHK 9901034, KNP 0501060]
Parity	$n/2$	[D 9805006]
Symmetric functions	$\sqrt{(n(n-\Gamma))}$	[BBCMW 9802049]
Most functions	$\Theta(n)$	[A 9811080]
Graph connectivity	$n^{3/2}$	[DHHM 0401091]
Ordered search	$\Theta(\log n)$	[BH 0703231, CLP 0608161]


∧
∨
∨
x_1
x_2
∧
∨
∨
x_3
. . .
x_n

OR	$\sqrt{n}$	[G 9605043, BBBV 9701001]
Hidden subgroup	$\log \lvert G \rvert$	[EHK 9901034, KNP 0501060]
Parity	$n/2$	[D 9805006]
Symmetric functions	$\sqrt{(n(n-\Gamma))}$	[BBCMW 9802049]
Most functions	$\Theta(n)$	[A 9811080]
Graph connectivity	$n^{3/2}$	[DHHM 0401091]
Ordered search	$\Theta(\log n)$	[BH 0703231, CLP 0608161]
AND-OR formula	$\sqrt{n}$	[FGG 0702144, ACRSZ 0703015, R 0907.1623]

Element distinctness

 $\in \Sigma^n$

OR	$\sqrt{n}$	[G 9605043, BBBV 9701001]
Hidden subgroup	$\log \lvert G \rvert$	[EHK 9901034, KNP 0501060]
Parity	$n/2$	[D 9805006]
Symmetric functions	$\sqrt{(n(n-\Gamma))}$	[BBCMW 9802049]
Most functions	$\Theta(n)$	[A 9811080]
Graph connectivity	$n^{3/2}$	[DHHM 0401091]
Ordered search	$\Theta(\log n)$	[BH 0703231, CLP 0608161]
AND-OR formula	$\sqrt{n}$	[FGG 0702144, ACRSZ 0703015, R 0907.1623]
Collision	$n^{1/3}$	[BHT 9705002, AS 04]
Element distinctness	$n^{2/3}$	[A 0311001, MSS 0310134, S 0401053]

a subset of size k of entries satisfying some property

Given $x \in \Sigma^n$, find ~~k equal entries~~, if possible

Algorithm:

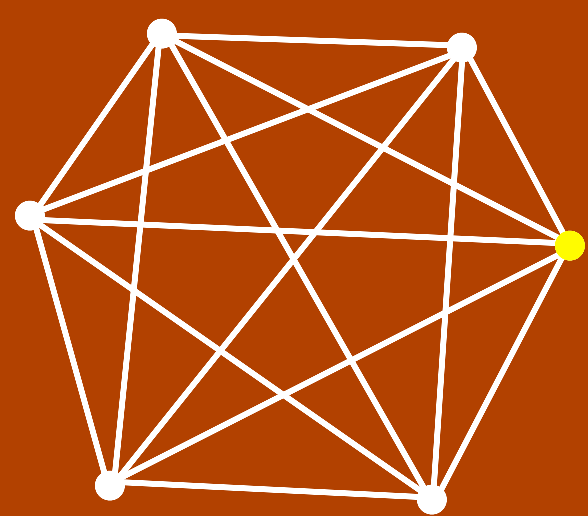
1. Query, and remember, a random r positions of x
   - r = "database size"
   - Check for a good subset—probability $\binom{n-k}{r-k} / \binom{n}{r} \approx (r/n)^k$
2. Repeat $(n/r)^{k/2}$ times:
   a. Add a -1 phase if the database includes a good subset
   b. $\sqrt{r}$ times: Move to an adjacent subset by a quantum walk (one query to add & one to delete an element)
3. Measure the subset and check it

$$Q \leq r + (n/r)^{k/2} \times \sqrt{r} = n^{k/(k+1)} \text{ for } r = n^{k/(k+1)}$$

[A 0311001, MSS 0310134, CE 0311038, S 0401053]

Algorithm:

1. Query, and remember, a random r positions of x
   - r = "database size"
   - Check for a good subset—probability $\binom{n-k}{r-k} / \binom{n}{r} \approx (r/n)^k$
2. Repeat $(n/r)^{k/2}$ times:
   a. Add a -1 phase if the database includes a good subset
   b. $\sqrt{r}$ times: Move to an adjacent subset by a quantum walk (one query to add & one to delete an element)



subsets of size r

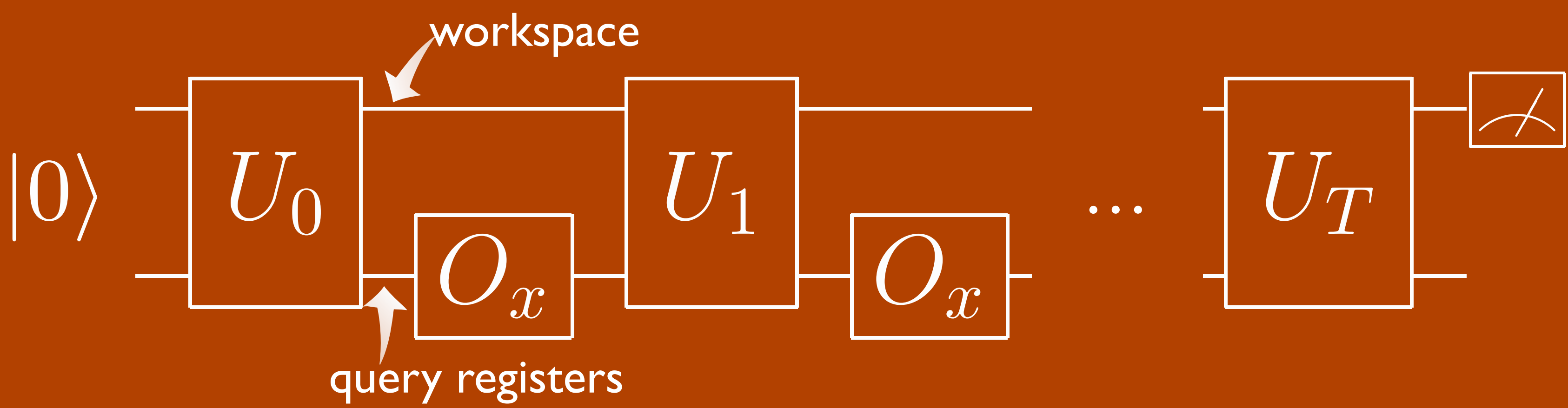
OR	$\sqrt{n}$	[G 9605043, BBBV 9701001]
Hidden subgroup	$\log \lvert G \rvert$	[EHK 9901034, KNP 0501060]
Parity	$n/2$	[D 9805006]
Symmetric functions	$\sqrt{(n(n-\Gamma))}$	[BBCMW 9802049]
Most functions	$\Theta(n)$	[A 9811080]
Graph connectivity	$n^{3/2}$	[DHHM 0401091]
Ordered search	$\Theta(\log n)$	[BH 0703231, CLP 0608161]
AND-OR formula	$\sqrt{n}$	[FGG 0702144, ACRSZ 0703015, R 0907.1623]
Collision	$n^{1/3}$	[BHT 9705002, AS 04]
Element distinctness	$n^{2/3}$	[A 0311001, MSS 0310134, S 0401053]
Triangle finding	$(n, n^{1.3})$	[MSS 0310134, CE 0311038, MNRS 0608026]
Matrix product verification	$\leq n^{5/3}$	[BS 0409035]
Hamiltonian simulation	$(\sqrt{n}, n^{2/3})$	[BC 0910.4157]
Bdd-degree bipartite property	$n^{1/3}$	[ACL 1012.3174]
Graph planarity	$n^{3/2}$	[CK 1011.1443 **Tuesday**]
State generation: Index erasure	$\sqrt{n}$	[AMRR 1012.2112 **Tuesday**]

Most of these algorithms are also time efficient

Quantum query complexity lower bounds:

- Polynomial method

$$Q_\epsilon(f) \geq \tfrac{1}{2}\deg_\epsilon(f)$$



$$\begin{aligned} O_x &= \sum_j (-1)^{x_j} |j\rangle\langle j| \\ &= \sum_j (1 - 2x_j)|j\rangle\langle j| \end{aligned}$$

[BBCMW 9802049]

Quantum query complexity lower bounds:

- Polynomial method

$$Q_\epsilon(f) \geq \tfrac{1}{2}\deg_\epsilon(f)$$

Can be *very* loose:

For OR on $\{0^n, 10^{n-1}, \ldots, 0^{n-1}1\}$,
$f(x)=x_1+x_2+\ldots+x_n$ so $\deg(f)=1$
but $Q(f) = \Theta(\sqrt{n})$

For *total* functions on $\{0,1\}^n$:

$$Q_\epsilon(f) = O(\deg_\epsilon(f)^6)$$

largest known separation is

$$Q(f) = \Omega(\deg(f)^{1.3})$$

- Adversary method

$$Q_\epsilon(f) \geq \frac{1-2\sqrt{\epsilon(1-\epsilon)}}{2}\mathrm{Adv}(f)$$

Also,

- General adversary bound
- Multiplicative adversary

[BBCMW 9802049]

A certificate for input x is a set of positions whose values fix f.

For f=OR:

Input	Minimal certificate
00110	{3}
00000	{1,2,3,4,5}

$\Rightarrow C(OR_n)=n$

Given a certificate for the input, it suffices to read those bits

$\therefore$ Certificate complexity = Nondeterministic query complexity

For $f : \{0,1\}^n \to \{0,1\}$,

$$f(x) = 0,\ f(y) = 1$$

$$\Rightarrow \quad c_x \cap c_y \neq \emptyset$$

For $f : \{0,1\}^n \to \{0,1\}$,

$$f(x) = 0,\ f(y) = 1$$

$$\Rightarrow \quad (c_x \cap c_y) \cap \{j : x_j \neq y_j\} \neq \emptyset$$

$$C(f) = \min_{\text{certificates } c_x} \max_x |c_x|$$

$$C(f) = \min_{\{\vec{p}_x \in \{0,1\}^n\}} \max_x \sum_j p_x[j]$$

$$\text{s.t.} \sum_{j: x_j \neq y_j} p_x[j] p_y[j] \geq 1 \quad \text{if } f(x) \neq f(y)$$

$$\mathrm{Adv}(f) = \min_{\{\vec{p}_x \in \mathbb{R}^n\}} \max_x \sum_j p_x[j]^2$$

$$\text{s.t.} \sum_{j: x_j \neq y_j} p_x[j] p_y[j] \geq 1 \ \text{ if } f(x) \neq f(y)$$

$$\mathrm{Adv}(f) = \min_{\{\vec{p}_x \in \mathbb{R}^n\}} \max_x \|\vec{p}_x\|^2$$

$$\text{s.t.} \sum_{j: x_j \neq y_j} p_x[j] p_y[j] \geq 1 \ \text{ if } f(x) \neq f(y)$$

For OR on $\{0^n, 10^{n-1}, \ldots, 0^{n-1}1\}$,

x	0^n	10^{n-1}	$\ldots$	$0^{n-1}1$
$\vec{p}_x$	$(1, \ldots, 1)$	$(1, 0, \ldots, 0)$		$(0, \ldots, 0, 1)$

$$\Rightarrow \mathrm{Adv}(\mathrm{OR}_n) = \sqrt{n}$$

$\mathrm{Adv}(f) \leq \sqrt{C_0(f) C_1(f)}$ for total functions

$\mathrm{Adv}(f) \leq \sqrt{n \min\{C_0(f), C_1(f)\}}$ in general

ED

Problem 2. This is still a minimization problem

$$\mathrm{Adv}(f) = \min_{\{\vec{p}_x \in \mathbb{R}^n\}} \max_x \|\vec{p}_x\|^2$$

$$\text{s.t.} \sum_{j : x_j \neq y_j} p_x[j] p_y[j] \geq 1 \quad \text{when } f(x) \neq f(y)$$

Take the dual:

$$\mathrm{Adv}(f) = \max_{\Gamma \in \mathbb{R}^{\mathcal{D} \times \mathcal{D}}} \|\Gamma\|$$

$$\text{s.t.} \ \Gamma[x,y] \geq 0$$

$$\Gamma[x,y] = 0 \text{ if } f(x) = f(y)$$

$$\forall j \ \left\| \Gamma \circ \sum_{x,y : x_j \neq y_j} |x\rangle\langle y| \right\| \leq 1$$

$$Q(f) = \Omega(\mathrm{Adv}(f))$$

$$\begin{tikzcd} \underline{t=0} & \underline{t=1} & \underline{t=2} \\ & O_x|\psi\rangle \arrow[r] & O_xU_1O_x|\psi\rangle \\ |\psi\rangle \arrow[ur] \arrow[dr] & & \\ & O_y|\psi\rangle \arrow[r] & O_yU_1O_y|\psi\rangle \end{tikzcd}$$

Idea: Track the divergence of pairs (x,y) with $f(x) \neq f(y)$

All pairs can't diverge at once,

e.g., querying bit 1 only separates pairs with $x_1 \neq y_1$

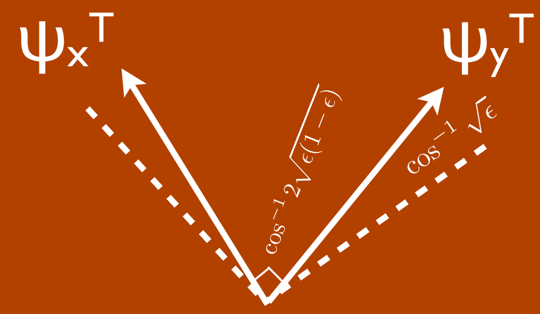
Gram matrix $\rho^t[x,y]=\langle\psi_x^t, \psi_y^t\rangle$

<u>Initially</u>

ρ^0 = all-ones matrix (J)

<u>Finally</u>

$|\rho^T[x,y]| \leq 2\sqrt{(\varepsilon(1-\varepsilon))}$



Proof: For a *D*×*D* matrix M, let

$$A(M) = \min_{\{\vec{p}_x \in \mathbb{R}^n\}} \max_x \|\vec{p}_x\|^2$$

$$\sum_{j:x_j \neq y_j} p_x[j]p_y[j] \geq |M[x,y]|$$

- A(M+N)≤A(M)+A(N)
- A(M)≤A(N) if M≤N entry-wise
- If J=all-ones matrix, F[x,y]=$\delta_{f(x),f(y)}$,
  Adv(f)=A(J-F)

$$\begin{array}{cc} & \begin{array}{ccc} f^{-1}(0) & f^{-1}(1) & f^{-1}(2) \end{array} \\ \begin{array}{c} f^{-1}(0) \\ f^{-1}(1) \\ f^{-1}(2) \end{array} & \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \end{array}$$

Proof: For a *D*×*D* matrix M, let

$$A(M) = \min_{\{\vec{p}_x \in \mathbb{R}^n\}} \max_x \|\vec{p}_x\|^2$$

$$\sum_{j:x_j \neq y_j} p_x[j] p_y[j] \geq |M[x,y]|$$

- A(M+N)≤A(M)+A(N)
- A(M)≤A(N) if M≤N entry-wise
- If J=all-ones matrix, $F[x,y]=\delta_{f(x),f(y)}$, Adv(f)=A(J-F)

(a distance: ρ^0=J, "ρ^T almost lies under F")

$$\mathrm{Adv}(f) = A(J-F) = A(\rho^0 \circ (J-F))$$

$$= A\left(\left(\rho^T + \sum_{t=0}^{T-1} (\rho^t - \rho^{t+1})\right) \circ (J-F)\right)$$

$$\leq A(\rho^T \circ (J-F)) + \sum_{t=0}^{T-1} A(\rho^t - \rho^{t+1})$$

Proof: For a *D*×*D* matrix M, let

$$A(M) = \min_{\{\vec{p}_x \in \mathbb{R}^n\}} \max_x \|\vec{p}_x\|^2$$

$$\text{s.t.} \sum_{j: x_j \neq y_j} p_x[j] p_y[j] \geq |M[x,y]|$$

- $A(M+N) \leq A(M) + A(N)$
- $A(M) \leq A(N)$ if $M \leq N$ entry-wise
- If J=all-ones matrix, $F[x,y]=\delta_{f(x),f(y)}$,
  Adv(f)=A(J-F)

(a distance: ρ^0=J, "ρ^T almost lies under F")

$$\mathrm{Adv}(f) = A(J-F) \leq \underbrace{A(\rho^T \circ (J-F))}_{\leq\, 2\sqrt{\epsilon(1-\epsilon)}A(J-F)} + \sum_{t=0}^{T-1} \underbrace{A(\rho^t - \rho^{t+1})}_{\leq\, 2}$$

1. ψ_x^T (1-ε)-distinguishable from ψ_y^T $\Rightarrow |\langle \psi_x^T | \psi_y^T \rangle| \leq 2\sqrt{\epsilon(1-\epsilon)}$
2. Each step is small…

⇒ T is large: $\mathrm{Adv}(f) \leq 2\sqrt{\epsilon(1-\epsilon)}\mathrm{Adv}(f) + 2T$ $\square$

Each step is small: $A(\rho^t - \rho^{t+1}) \leq 2$

t=0 t=1

$$\begin{tikzcd}
 & O_x|\psi\rangle \\
|\psi\rangle \arrow[ur, "\text{input x}"] \arrow[dr, "\text{input y}"'] & \\
 & O_y|\psi\rangle
\end{tikzcd}$$

$$(\rho^0 - \rho^1)[x,y] = \langle\psi|\psi\rangle - \langle\psi|O_x^\dagger O_y|\psi\rangle = 2\sum_{j:x_j \neq y_j} \langle\psi|j\rangle\langle j|\psi\rangle$$

$$O_x^\dagger O_y = \sum_j (-1)^{x_j + y_j} |j\rangle\langle j| = I - 2\sum_{j:x_j \neq y_j} |j\rangle\langle j|$$

General adversary bound

$\mathrm{Adv}^{\pm}$

$$C(f) = \min_{\{\vec{p}_x \in \{0,1\}^n\}} \max_x \|\vec{p}_x\|^2$$

$$\text{s.t.} \sum_{j: x_j \neq y_j} p_x[j] p_y[j] \geq 1 \quad \text{if } f(x) \neq f(y)$$

$$\text{Adv}(f) = \min_{\{\vec{p}_x \in \mathbb{R}^n\}} \max_x \|\vec{p}_x\|^2$$

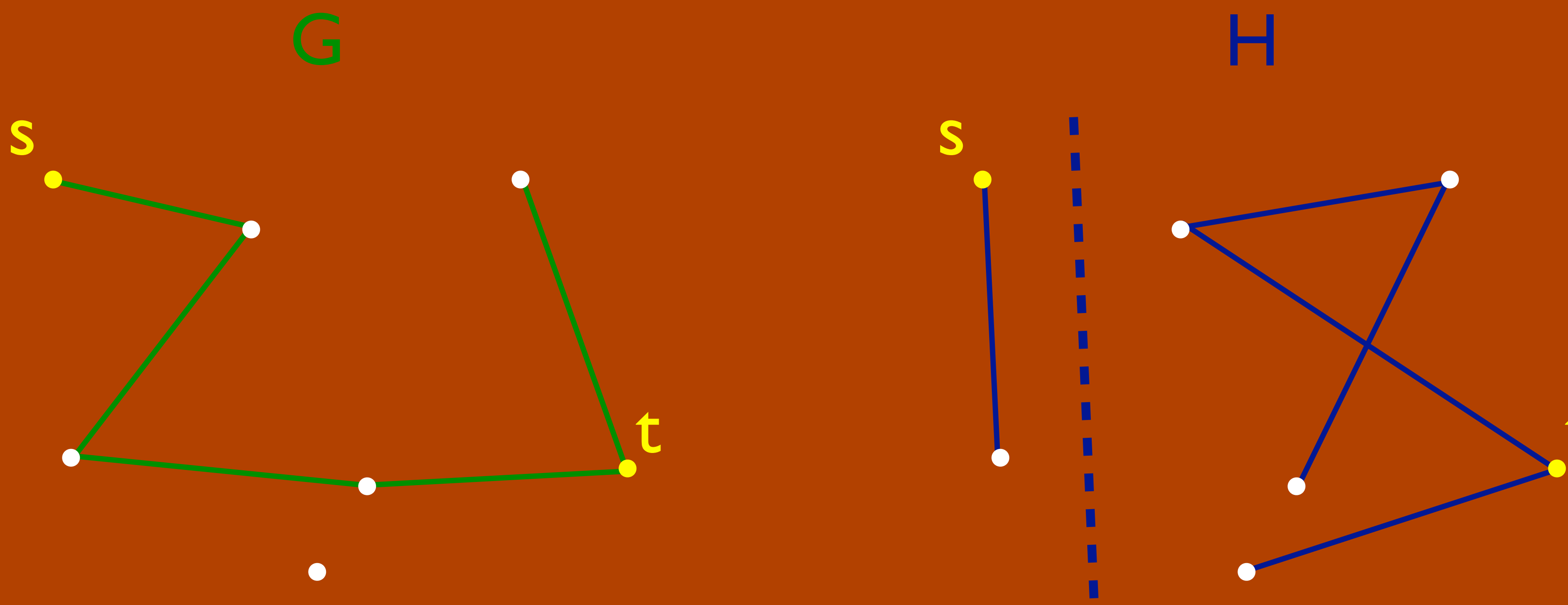
$$\text{s.t.} \sum_{j: x_j \neq y_j} p_x[j] p_y[j] \geq 1 \quad \text{if } f(x) \neq f(y)$$

$$
\begin{aligned}
C(f) = \min_{\{p_{xj} \in \{0,1\}\}} \max_x \sum_j p_{xj}^2 \\
\text{s.t.} \sum_{j: x_j = y_j} p_{xj} p_{yj} \geq 1 \quad \text{if } f(x) \neq f(y)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Adv}(f) = \min_{\{p_{xj} \in \mathbb{R}\}} \max_x \sum_j p_{xj}^2 \\
\text{s.t.} \sum_{j: x_j = y_j} p_{xj} p_{yj} \geq 1 \quad \text{if } f(x) \neq f(y)
\end{aligned}
$$

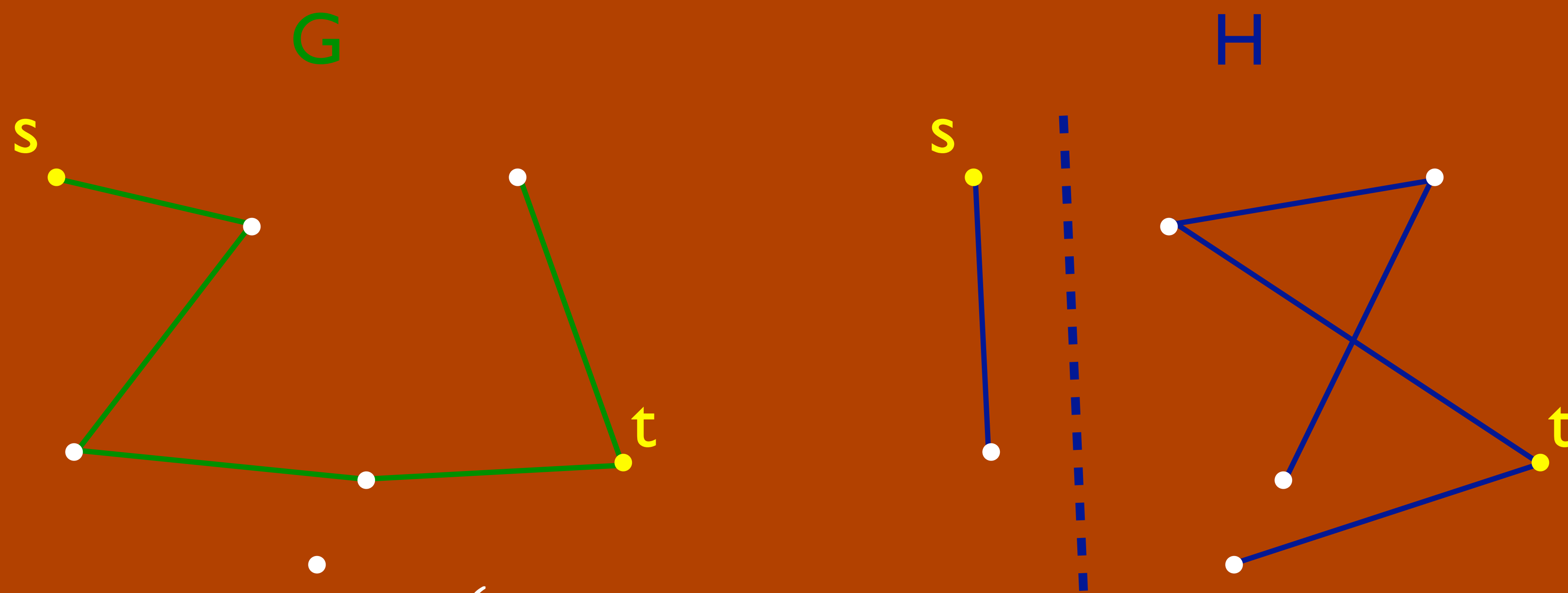
$$
\begin{aligned}
\mathrm{Adv}^{\pm}(f) = \min_{\{|u_{xj}\rangle \in \mathbb{R}^m\}} \max_x \sum_j \| u_{xj} \|^2 \\
\text{s.t.} \sum_{j: x_j \neq y_j} \langle u_{xj} | u_{yj} \rangle = 1 \quad \text{if } f(x) \neq f(y)
\end{aligned}
$$

Example: s-t Connectivity



$p_{Ge} = 1$ along the path, $p_{He} = 1$ across the cut

$$\Rightarrow \sum_{e:e\in G, e\notin H} p_{Ge}p_{He} \geq 1 \qquad \Rightarrow \text{Adv}(f) \leq \sqrt{dn^2} \leq n^{3/2}$$



$$u_{Ge} = \begin{cases} 1 & \text{path crosses } e \text{ to the right} \\ -1 & \text{path crosses } e \text{ to the left} \end{cases}$$

$u_{He} = 1$ across the cut

$$\Rightarrow \sum_{e: e \in G, e \notin H} \langle u_{Ge} | u_{He} \rangle = (\#\text{ right crossings}) - (\#\text{ left crossings}) = 1$$

$$\Rightarrow \mathrm{Adv}^{\pm}(f) \leq n^{3/2}$$

Why is Adv$^{\pm}$ a semi-definite program (SDP)?

$$\min_{X\succcurlyeq 0} \mathrm{Tr}(C^T X)$$
$$\text{s.t. } \mathrm{Tr}(A_i^T X) = a_i$$

$$\max_{\{b_i\}} \sum\nolimits_i a_i b_i$$
$$\text{s.t. } C - \sum\nolimits_i b_i A_i \succcurlyeq 0$$

$$\mathrm{Adv}^{\pm}(f) = \min_{\{u_{xj}\in\mathbb{R}^m\}} \max_x \sum\nolimits_j \|u_{xj}\|^2$$
$$\text{s.t. } \sum_{j:x_j\neq y_j} \langle u_{xj}|u_{yj}\rangle{=}1 \quad \text{if } f(x)\neq f(y)$$

$$X_j[x,y] = \langle u_{xj}|u_{yj}\rangle$$

$$\mathrm{Adv}^{\pm}(f) = \min_{\{X_j\succcurlyeq 0\}} \max_x \sum_j \langle x|X_j|x\rangle$$
$$\text{s.t. } \sum_{j:x_j\neq y_j} \langle x|X_j|y\rangle = 1 \;\; \text{if } f(x)\neq f(y)$$

$$\mathrm{Adv}^{\pm}(f) = \max_{\Gamma \in \mathbb{R}^{\mathcal{D}\times\mathcal{D}}} \|\Gamma\|$$

subject to:

$$\cancel{\Gamma[x,y] \geq 0}$$

$$\Gamma[x,y] = 0 \text{ if } f(x) = f(y)$$

$$\forall j \left\| \Gamma \circ \sum_{x,y:x_j \neq y_j} |x\rangle\langle y| \right\| \leq 1$$

General adversary bound:

For f: $D \rightarrow \{0,1\}$:

$$\mathrm{Adv}^{\pm}(f) = \min_{\{u_{xj} \in \mathbb{R}^m\}} \max_x \sum_j \|u_{xj}\|^2$$

$$\text{s.t.} \sum_{j:x_j \neq y_j} \langle u_{xj} | u_{yj} \rangle = 1 \quad \text{if } f(x) \neq f(y)$$

For f: $D \rightarrow E$:

$$\mathrm{Adv}^{\pm}(f) = \min_{\{u_{xj}, v_{xj} \in \mathbb{R}^m\}} \max_x \max \left\{ \sum_j \|u_{xj}\|^2, \sum_j \|v_{xj}\|^2 \right\}$$

$$\text{s.t.} \sum_{j:x_j \neq y_j} \langle u_{xj} | v_{yj} \rangle = 1 \quad \text{if } f(x) \neq f(y)$$

$$Q(f) = \Omega(\mathrm{Adv}^{\pm}(f))$$

$$A^{\pm}(M) = \min_{\{\vec{u}_{xj} \in \mathbb{R}^m\}} \max_x \sum_j \|u_{xj}\|^2$$
$$\text{s.t.} \sum_{j : x_j \neq y_j} \langle u_{xj} | u_{yj} \rangle = M[x, y]$$

- Same initial condition
- Same single-step factorization
- More involved final case

[HLS 0611054]

A. Query complexity

B. Adversary lower bounds

C. Spectra of reflections

D. Adversary upper bound

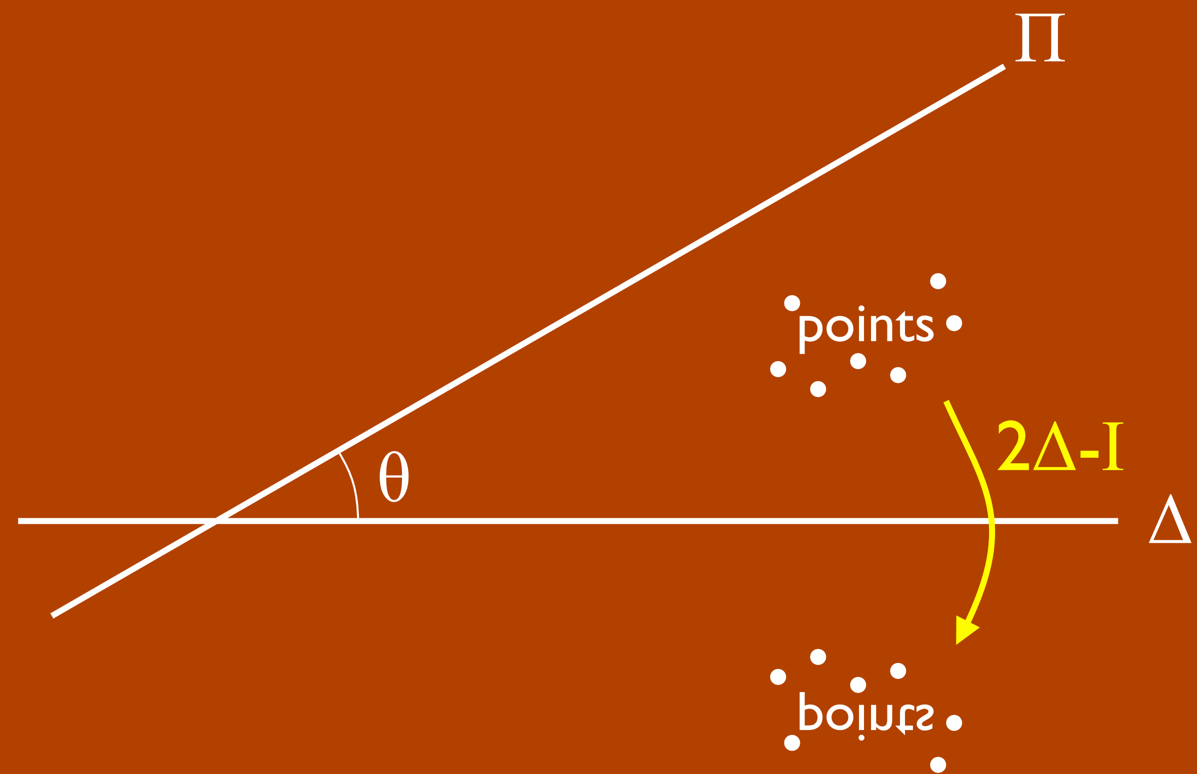
$$Q(f) = \Theta(\mathrm{Adv}^{\pm}(f))$$

A reflection is an operator that squares to the identity.

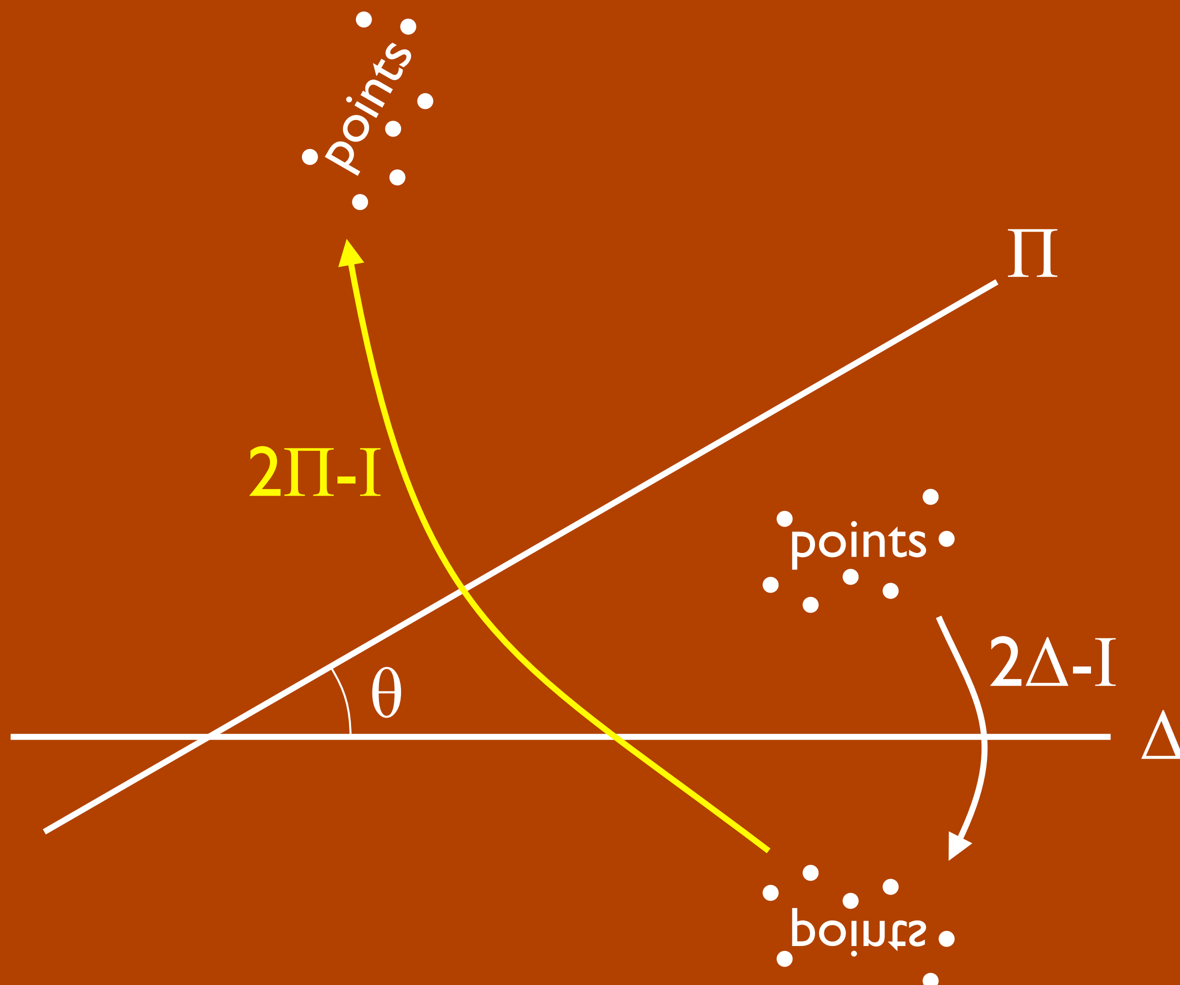
A reflection is a unitary with eigenvalues ± 1.

$$R = \Pi - (I - \Pi) = 2\Pi - I$$

Π = projection onto the +1-eigenvalue eigenspace of R

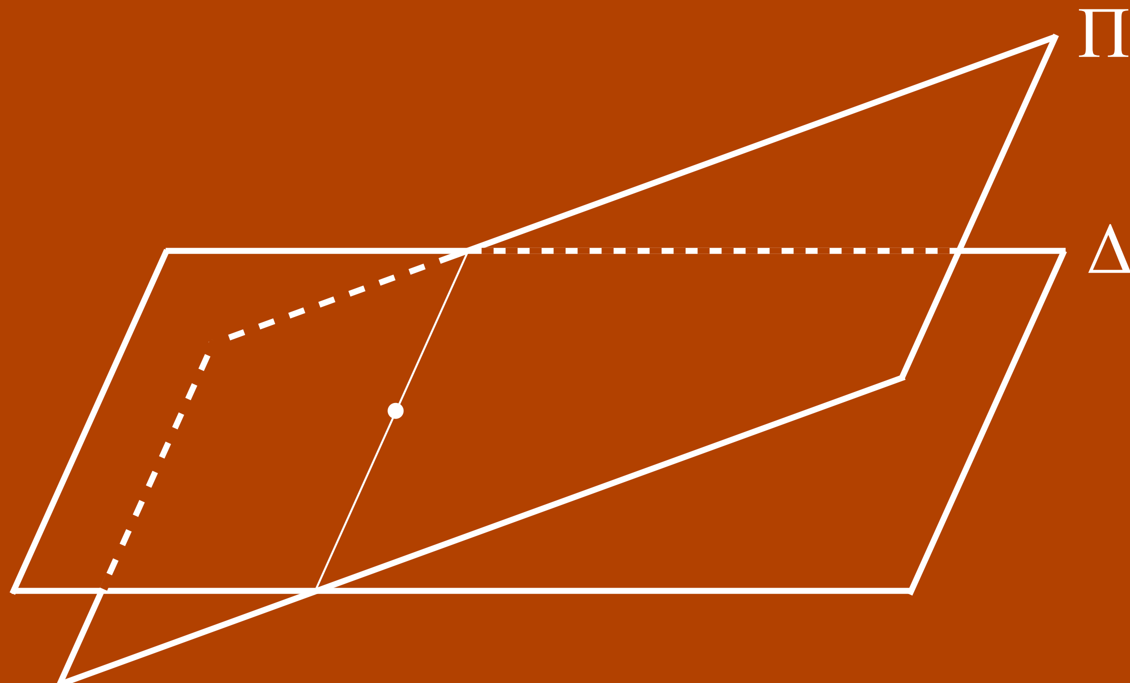


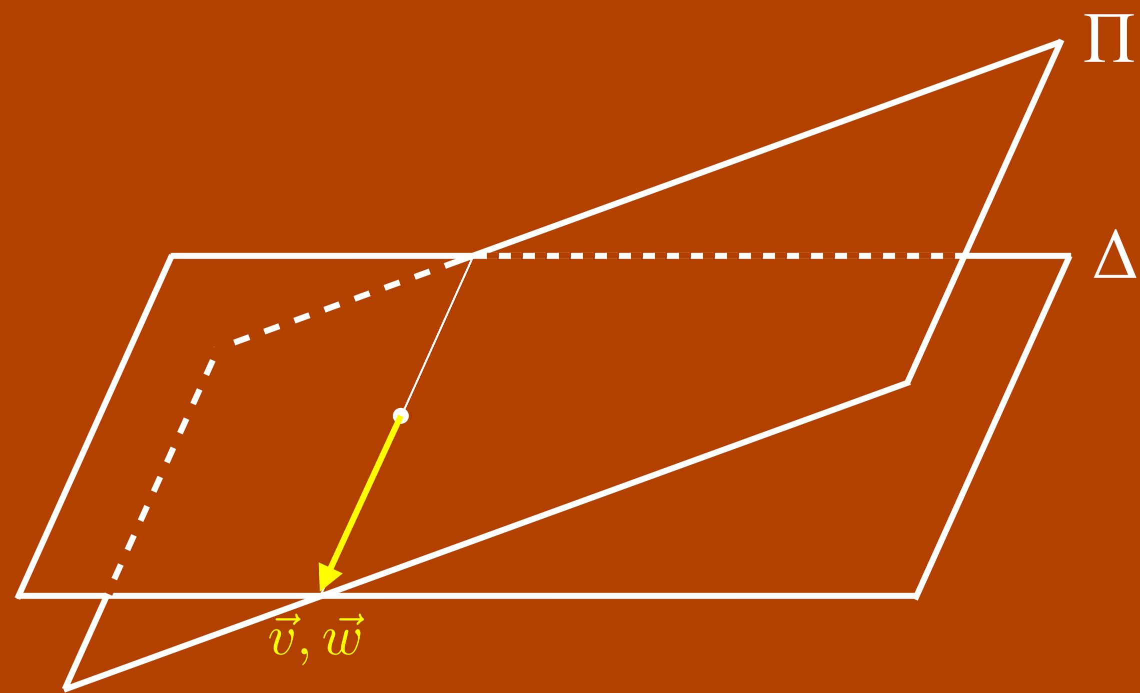
Π
points
θ
2Δ-I
Δ
points



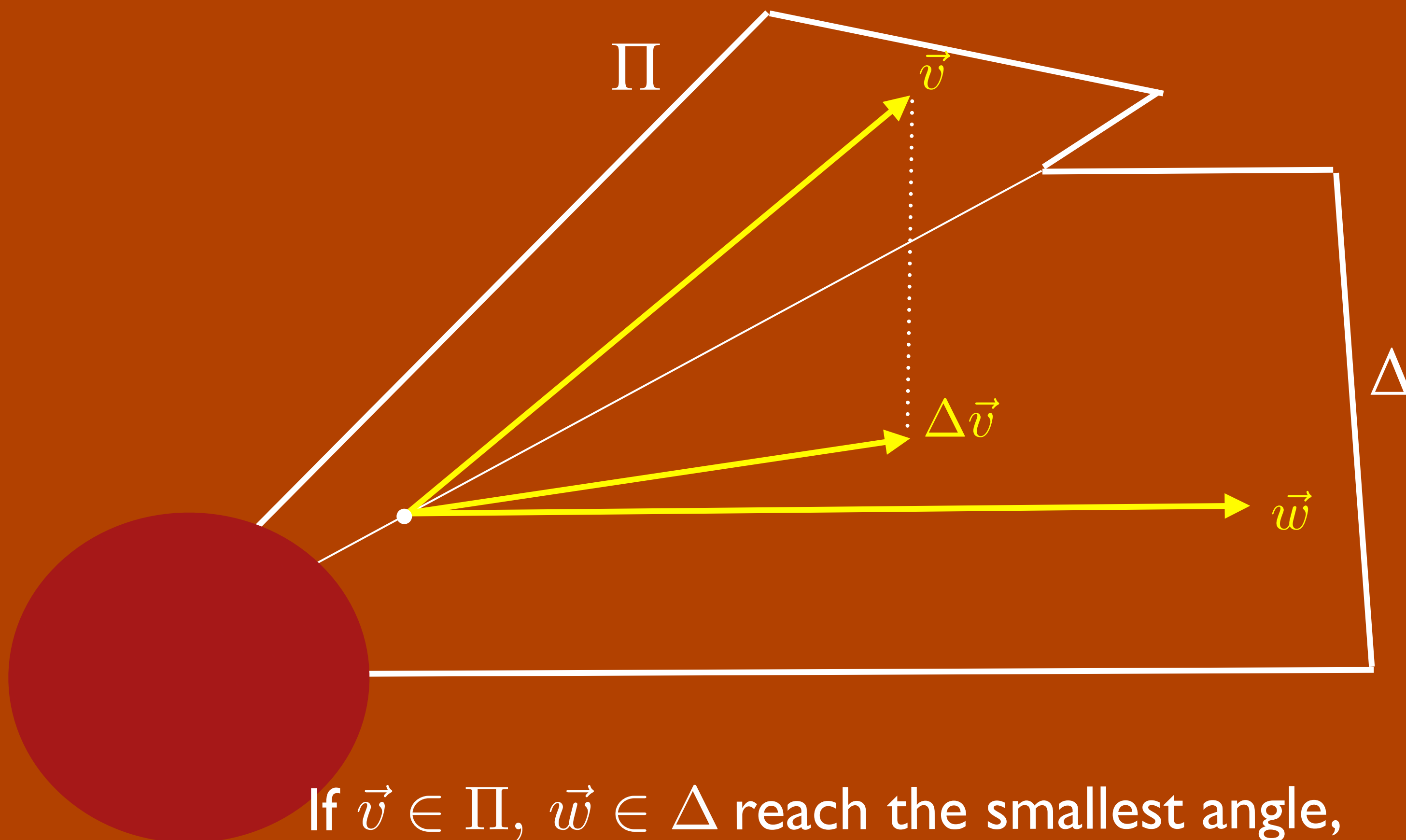
(2Π-I)(2Δ-I) is a rotation by angle 2θ,
eigenvalues $e^{\pm 2i\theta}$, eigenvectors (1,±i)

Two subspaces will not generally lie at a fixed angle





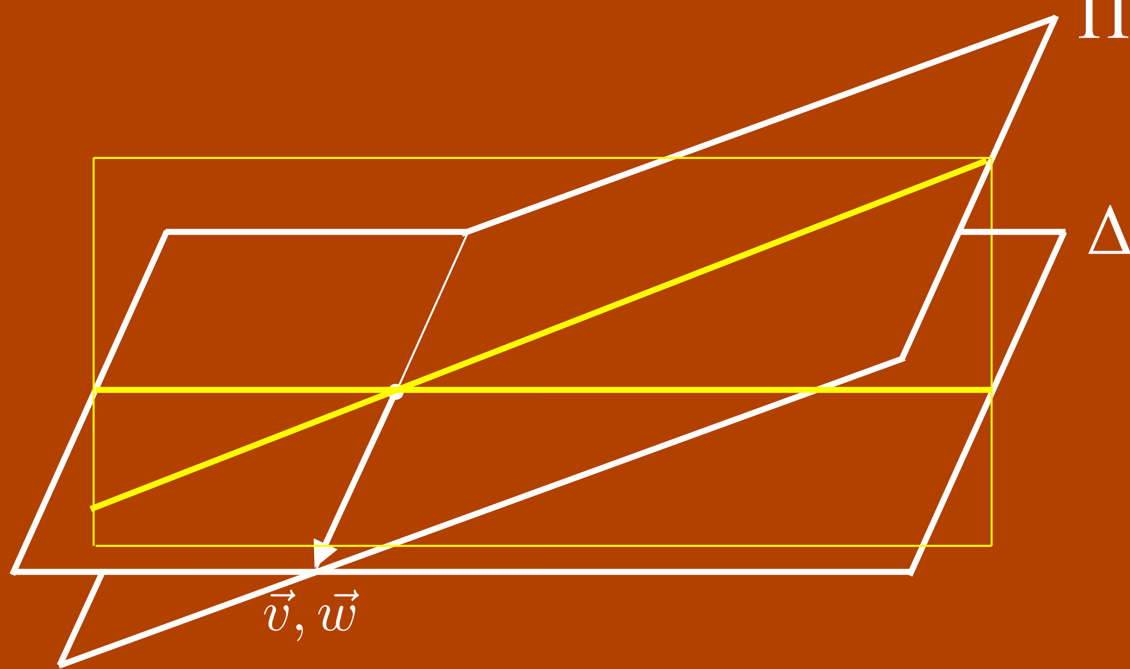
If $\vec{v} \in \Pi$, $\vec{w} \in \Delta$ reach the smallest angle,
then $\Delta\vec{v} \propto \vec{w}$, $\Pi\vec{w} \propto \vec{v}$



If $\vec{v} \in \Pi$, $\vec{w} \in \Delta$ reach the smallest angle,
then $\Delta\vec{v} \propto \vec{w}$, $\Pi\vec{w} \propto \vec{v}$

$$\Delta\vec{v} = \lambda\vec{w} + \mu\vec{w}^{\perp}$$

$$\Rightarrow \frac{1}{\sqrt{1+\epsilon^2}}\langle v, w + \epsilon w^{\perp}\rangle = \frac{\langle v, w\rangle + \epsilon\mu}{\sqrt{1+\epsilon^2}} > \langle v, w\rangle$$



If $\vec{v} \in \Pi$, $\vec{w} \in \Delta$ reach the smallest angle,
then $\Delta\vec{v} \propto \vec{w}$, $\Pi\vec{w} \propto \vec{v}$

$\therefore$ Span$\{\vec{v}, \vec{w}\}$ is fixed by Π and Δ

$\therefore$ Span$\{\vec{v}, \vec{w}\}^{\perp}$ is fixed by Π and Δ

Jordan's Lemma (1875)

Two reflections acting on a Hilbert space decompose it into irreducible one- and two-dimensional subspaces

Any two projections can be simultaneously block-diagonalized with blocks of dimension at most two

$$U\Pi U^\dagger = \left(\begin{array}{c:c:c} \ddots & 0 & 0 \\ \hdashline 0 & \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} & 0 \\ \hdashline 0 & 0 & \ddots \end{array}\right) \qquad U\Delta U^\dagger = \left(\begin{array}{c:c:c} \ddots & 0 & 0 \\ \hdashline 0 & \begin{smallmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{smallmatrix} & 0 \\ \hdashline 0 & 0 & \ddots \end{array}\right)$$

Up to an isometry,

$$(2\Pi - I)(2\Delta - I) = \sum_\beta |\beta\rangle\langle\beta| \otimes \begin{pmatrix} \cos 2\theta_\beta & -\sin 2\theta_\beta \\ \sin 2\theta_\beta & \cos 2\theta_\beta \end{pmatrix}$$

In-place amplification of QMA

Quantum zero knowledge

Szegedy correspondence

Discrete-time quantum walks (unitary)

Continuous-time quantum walks (Hamiltonian)

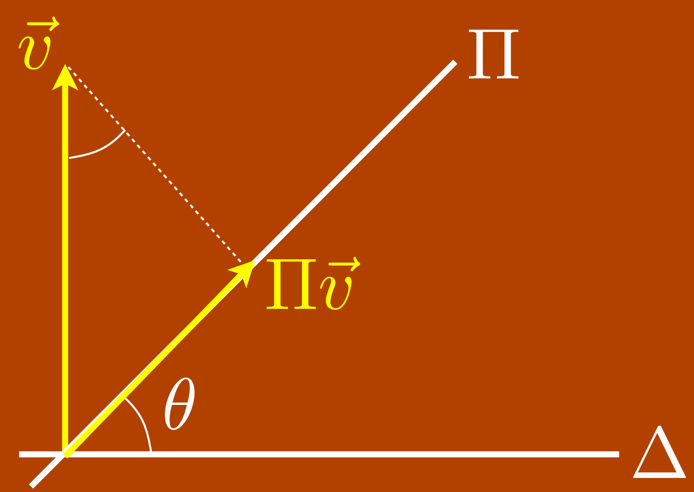
Random walks

1. [MW cs/0506068, NWZ 0904.1549]
2. [W 0511020]
3. [S 0401053, MNRS 0608026, RS 0710.2630, C 0810.0312]

Effective Spectral Gap Lemma:

- Let Π, Δ be two projections
- Let P_Θ be the projection onto eigenvectors of $(2\Pi\text{-}1)(2\Delta\text{-}1)$ with phase less than 2Θ in magnitude
- Then for any $\vec{v}$ with $\Delta\vec{v} = 0$,

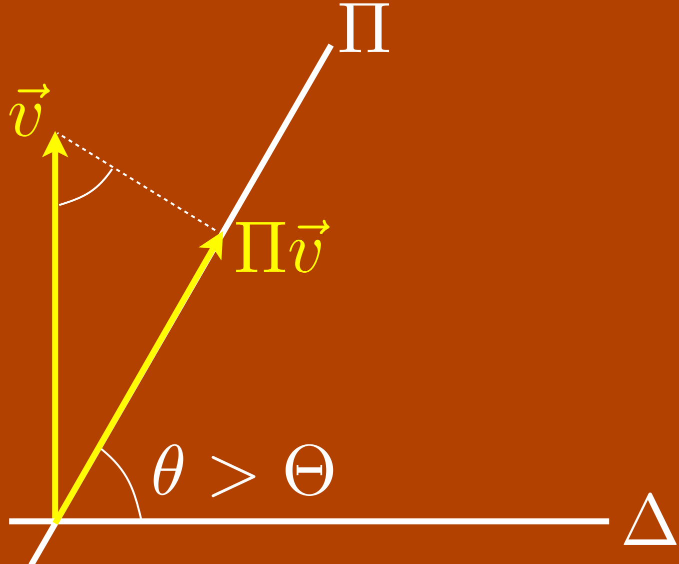
$$\|P_\Theta \Pi \vec{v}\| \leq \Theta \|\vec{v}\|$$



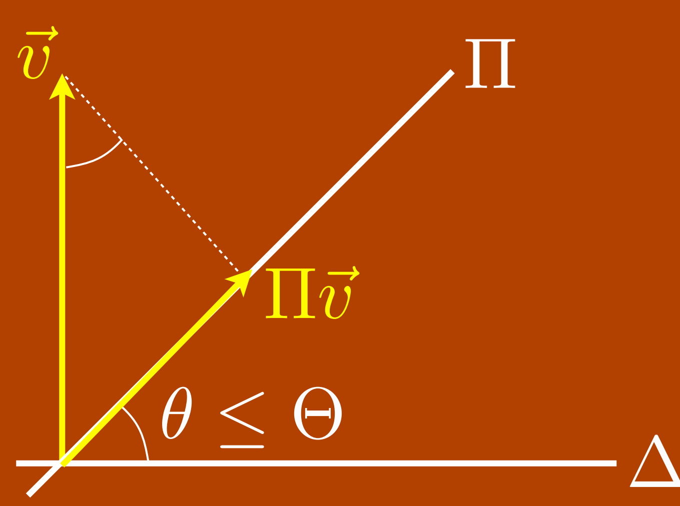
Effective Spectral Gap Lemma:

- Let Π, Δ be two projections
- Let P_Θ be the projection onto eigenvectors of $(2\Pi\text{-}1)(2\Delta\text{-}1)$ with phase less than 2Θ in magnitude
- Then for any $\vec{v}$ with $\Delta\vec{v} = 0$,

$$\|P_\Theta \Pi \vec{v}\| \leq \Theta \|\vec{v}\|$$



$$P_\Theta \Pi \vec{v} = 0$$



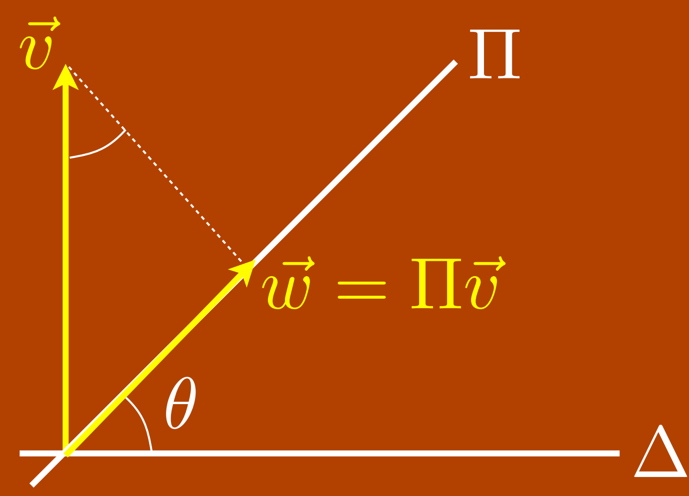
$$P_\Theta \Pi \vec{v} = \Pi \vec{v}$$

Effective Spectral Gap Lemma:

- Let Π, Δ be two projections
- Let P_Θ be the projection onto eigenvectors of $(2\Pi-1)(2\Delta-1)$ with phase less than 2Θ in magnitude
- Then for any $\vec{v}$ with $\Delta\vec{v} = 0$,

$$\|P_\Theta \Pi \vec{v}\| \leq \Theta \|\vec{v}\|$$

Application:



Effective Spectral Gap Lemma:

- Let Π, Δ be two projections
- Let P_Θ be the projection onto eigenvectors of $(2\Pi\text{-}1)(2\Delta\text{-}1)$ with phase less than 2Θ in magnitude
- Then for any $\vec{v}$ with $\Delta\vec{v} = 0$,

$$\|P_\Theta \Pi \vec{v}\| \leq \Theta \|\vec{v}\|$$

Proof: Jordan's Lemma $\Rightarrow$ Up to a change in basis,

$$\Delta = \sum_\beta |\beta\rangle\langle\beta| \otimes \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$$

$$\Pi = \sum_\beta |\beta\rangle\langle\beta| \otimes \begin{pmatrix} \cos^2\theta_\beta & \sin\theta_\beta\cos\theta_\beta \\ \sin\theta_\beta\cos\theta_\beta & \sin^2\theta_\beta \end{pmatrix}$$

$$\Delta|v\rangle = 0 \Rightarrow |v\rangle = \sum_\beta d_\beta |\beta\rangle \otimes \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$$

$$\Rightarrow P_\Theta \Pi |v\rangle = \sum_{\beta:|\theta_\beta|\leq\Theta} d_\beta |\beta\rangle \otimes \sin\theta_\beta \begin{pmatrix} \cos\theta_\beta \\ \sin\theta_\beta \end{pmatrix} \qquad \square$$

A. Query model

B. Adversary lower bounds

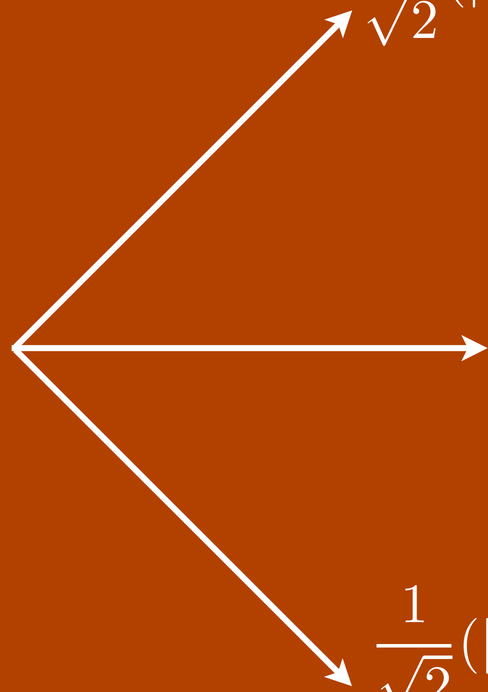
C. Spectra of reflections

D. Adversary upper bound

$$Q(f) = \Theta(\mathrm{Adv}^{\pm}(f))$$

For $f : \mathcal{D} \longrightarrow E$

let $\mathcal{H} = \mathbb{C}^{\{r\}} \oplus \mathbb{C}^{E}$, i.e., $\mathbb{C}^{1+|E|}$ with basis $|r\rangle$, $|e\rangle : e \in E$



The algorithm: Let $f : \{0,1\}^n \to E$

1. Begin with a vector solution to the SDP:

$$\min_{\{u_{xj}, v_{xj} \in \mathbb{C}^m\}} \quad \max_x \max \left\{ \sum_j \|u_{xj}\|^2, \sum_j \|v_{xj}\|^2 \right\}$$

$$\text{s.t. } \forall x, y \quad \sum_{j : x_j \neq y_j} \langle u_{xj} | v_{xj} \rangle = 1 - \delta_{f(x), f(y)}$$

$$\|$$

$$\min_{\substack{|\tilde{u}_x\rangle = \sum_j |j, x_j\rangle \otimes |u_{xj}\rangle \\ |\tilde{v}_y\rangle = \sum_j |j, \bar{y}_j\rangle \otimes |v_{yj}\rangle}} \quad \max_x \max \left\{ \|\tilde{u}_x\|^2, \|\tilde{v}_x\|^2 \right\}$$

$$\text{s.t. } \forall x, y \quad \langle \tilde{u}_x | \tilde{v}_y \rangle = 1 - \delta_{f(x), f(y)}$$

The algorithm: Let $f : \{0,1\}^n \to E$

1. Begin with a vector solution to the SDP:

$$W = \min_{\substack{|\tilde{u}_x\rangle = \sum_j |j, x_j\rangle \otimes |u_{xj}\rangle \\ |\tilde{v}_x\rangle = \sum_j |j, \bar{x}_j\rangle \otimes |v_{xj}\rangle}} \max_x \max \left\{ \|\tilde{u}_x\|^2, \|\tilde{v}_x\|^2 \right\}$$

$$\text{s.t. } \forall x, y \quad \langle \tilde{u}_x | \tilde{v}_y \rangle = 1 - \delta_{f(x), f(y)}$$

2. Let $\mathcal{H} = \mathbb{C}^{\{r\}} \oplus \mathbb{C}^E \oplus \left(\mathbb{C}^n \otimes \mathbb{C}^2 \otimes \mathbb{C}^m \right)$

$$|\psi_y\rangle = |r\rangle + |f(y)\rangle + \frac{1}{10\sqrt{W}} |\tilde{u}_y\rangle \in \mathcal{H}$$

$$\Delta = \text{projection onto span}\{\psi_y\}$$

$$\Pi_x = I - \sum_j |j\rangle\langle j| \otimes |\bar{x}_j\rangle\langle \bar{x}_j| \otimes I_m$$

3. Starting at r, alternate reflections about Δ and Π_x …

$$|\tilde{u}_x\rangle = \sum_j |j, x_j\rangle \otimes |u_{xj}\rangle$$
$$|\tilde{v}_x\rangle = \sum_j |j, \bar{x}_j\rangle \otimes |v_{xj}\rangle$$

$$|\psi_y\rangle = |r\rangle + |f(y)\rangle + \frac{1}{10\sqrt{W}}|\tilde{u}_y\rangle$$

$$\Delta = \text{proj. onto span}\{\psi_y\}$$

$$\Pi_x = I - \sum_j |j, \bar{x}_j\rangle\langle j, \bar{x}_j| \otimes I_m$$

$$\sqrt{W} = \max_x \max\left\{\|\tilde{u}_x\|, \|\tilde{v}_x\|\right\}$$

$$\langle \tilde{u}_x|\tilde{v}_y\rangle = 1 - \delta_{f(x),f(y)}$$

Lemma:

$$\vec{v} \in \Delta^\perp \Rightarrow \|P_\Theta \Pi \vec{v}\| \leq \Theta \|\vec{v}\|$$

The analysis:

$$|r\rangle = \tfrac{1}{2}\big(|r\rangle + |f(x)\rangle\big) + \tfrac{1}{2}\big(|r\rangle - |f(x)\rangle\big)$$

$|\tilde{u}_x\rangle = \sum_j |j, x_j\rangle \otimes |u_{xj}\rangle$

$|\tilde{v}_x\rangle = \sum_j |j, \bar{x}_j\rangle \otimes |v_{xj}\rangle$

$$|\psi_y\rangle = |r\rangle + |f(y)\rangle + \frac{1}{10\sqrt{W}}|\tilde{u}_y\rangle$$

$$\Delta = \text{proj. onto span}\{\psi_y\}$$

$$\Pi_x = I - \sum_j |j, \bar{x}_j\rangle\langle j, \bar{x}_j| \otimes I_m$$

$$\sqrt{W} = \max_x \max\left\{\|\tilde{u}_x\|, \|\tilde{v}_x\|\right\}$$

$$\langle \tilde{u}_x | \tilde{v}_y \rangle = 1 - \delta_{f(x), f(y)}$$

Lemma:

$$\vec{v} \in \Delta^{\perp} \Rightarrow \|P_{\Theta}\Pi\vec{v}\| \leq \Theta\|\vec{v}\|$$

The analysis:

$$|r\rangle = \frac{1}{2}\big(\underbrace{|r\rangle + |f(x)\rangle}_{\text{close to } |\psi_x\rangle}\big) + \frac{1}{2}\big(|r\rangle - |f(x)\rangle\big)$$

–doesn't move

$$|\tilde{u}_x\rangle = \sum_j |j, x_j\rangle \otimes |u_{xj}\rangle$$
$$|\tilde{v}_x\rangle = \sum_j |j, \bar{x}_j\rangle \otimes |v_{xj}\rangle$$

$$|\psi_y\rangle = |r\rangle + |f(y)\rangle + \frac{1}{10\sqrt{W}}|\tilde{u}_y\rangle$$

$$\Delta = \text{proj. onto span}\{\psi_y\}$$

$$\Pi_x = I - \sum_j |j, \bar{x}_j\rangle\langle j, \bar{x}_j| \otimes I_m$$

$$\sqrt{W} = \max_x \max \{\|\tilde{u}_x\|, \|\tilde{v}_x\|\}$$

$$\langle \tilde{u}_x|\tilde{v}_y\rangle = 1 - \delta_{f(x),f(y)}$$

Lemma:

$$\vec{v} \in \Delta^\perp \Rightarrow \|P_\Theta \Pi \vec{v}\| \leq \Theta \|\vec{v}\|$$

The analysis:

$$|r\rangle = \frac{1}{2}\underbrace{\big(|r\rangle + |f(x)\rangle\big)}_{\text{close to } |\psi_x\rangle \text{ –doesn't move}} + \frac{1}{2}\underbrace{\big(|r\rangle - |f(x)\rangle\big)}_{= \Pi_x\left(|r\rangle - |f(x)\rangle - 10\sqrt{W}|\tilde{v}_x\rangle\right)}$$

$$|\tilde{u}_x\rangle = \sum_j |j, x_j\rangle \otimes |u_{xj}\rangle$$
$$|\tilde{v}_x\rangle = \sum_j |j, \bar{x}_j\rangle \otimes |v_{xj}\rangle$$

$$|\psi_y\rangle = |r\rangle + |f(y)\rangle + \frac{1}{10\sqrt{W}} |\tilde{u}_y\rangle$$

$$\Delta = \text{proj. onto span}\{\psi_y\}$$

$$\Pi_x = I - \sum_j |j, \bar{x}_j\rangle\langle j, \bar{x}_j| \otimes I_m$$

$$\sqrt{W} = \max_x \max \{ \|\tilde{u}_x\|, \|\tilde{v}_x\| \}$$

$$\langle \tilde{u}_x | \tilde{v}_y \rangle = 1 - \delta_{f(x), f(y)}$$

Lemma:

$$\vec{v} \in \Delta^\perp \Rightarrow \|P_\Theta \Pi \vec{v}\| \leq \Theta \|\vec{v}\|$$

The analysis:

$$|r\rangle = \frac{1}{2} \underbrace{\Big(|r\rangle + |f(x)\rangle\Big)}_{\text{close to } |\psi_x\rangle \text{ –doesn't move}} + \frac{1}{2} \underbrace{\Big(|r\rangle - |f(x)\rangle\Big)}_{}$$

$$= \Pi_x \Big(|r\rangle - |f(x)\rangle - 10\sqrt{W} |\tilde{v}_x\rangle \Big)$$

$$\| \; \vec{v} \in \Delta^\perp$$

$$|\tilde{u}_x\rangle = \sum_j |j, x_j\rangle \otimes |u_{xj}\rangle$$
$$|\tilde{v}_x\rangle = \sum_j |j, \bar{x}_j\rangle \otimes |v_{xj}\rangle$$

$$|\psi_y\rangle = |r\rangle + |f(y)\rangle + \frac{1}{10\sqrt{W}} |\tilde{u}_y\rangle$$

$$\Delta = \text{proj. onto span}\{\psi_y\}$$

$$\Pi_x = I - \sum_j |j, \bar{x}_j\rangle\langle j, \bar{x}_j| \otimes I_m$$

$$\sqrt{W} = \max_x \max \{\|\tilde{u}_x\|, \|\tilde{v}_x\|\}$$

$$\langle \tilde{u}_x | \tilde{v}_y \rangle = 1 - \delta_{f(x), f(y)}$$

Lemma:
$$\vec{v} \in \Delta^\perp \Rightarrow \|P_\Theta \Pi \vec{v}\| \leq \Theta \|\vec{v}\|$$

The analysis:

$$|r\rangle = \underbrace{\tfrac{1}{2}\big(|r\rangle + |f(x)\rangle\big)}_{\text{close to } |\psi_x\rangle \text{ –doesn't move}} + \underbrace{\tfrac{1}{2}\big(|r\rangle - |f(x)\rangle\big)}_{= \Pi_x\left(|r\rangle - |f(x)\rangle - 10\sqrt{W}|\tilde{v}_x\rangle\right)}$$

$$\Pi_x\big(|r\rangle - |f(x)\rangle - 10\sqrt{W}|\tilde{v}_x\rangle\big) \qquad \big(|r\rangle - |f(x)\rangle - 10\sqrt{W}|\tilde{v}_x\rangle\big) = \vec{v} \in \Delta^\perp$$

$\Rightarrow$ $\Omega(1/W)$ effective spectral gap

$|\tilde{u}_x\rangle = \sum_j |j, x_j\rangle \otimes |u_{xj}\rangle$
$|\tilde{v}_x\rangle = \sum_j |j, \bar{x}_j\rangle \otimes |v_{xj}\rangle$

$$|\psi_y\rangle = |r\rangle + |f(y)\rangle + \frac{1}{10\sqrt{W}}|\tilde{u}_y\rangle$$

$$\Delta = \text{proj. onto span}\{\psi_y\}$$

$$\Pi_x = I - \sum_j |j, \bar{x}_j\rangle\langle j, \bar{x}_j| \otimes I_m$$

$$\sqrt{W} = \max_x \max\{\|\tilde{u}_x\|, \|\tilde{v}_x\|\}$$

$$\langle \tilde{u}_x | \tilde{v}_y \rangle = 1 - \delta_{f(x), f(y)}$$

Lemma:

$$\vec{v} \in \Delta^\perp \Rightarrow \|P_\Theta \Pi \vec{v}\| \le \Theta \|\vec{v}\|$$

The analysis:

$$|r\rangle = \underbrace{\tfrac{1}{2}\big(|r\rangle + |f(x)\rangle\big)}_{\text{close to } |\psi_x\rangle \text{ –doesn't move}} + \tfrac{1}{2}\underbrace{\big(|r\rangle - |f(x)\rangle\big)}$$

$$= \Pi_x\big(|r\rangle - |f(x)\rangle - 10\sqrt{W}|\tilde{v}_x\rangle\big)$$

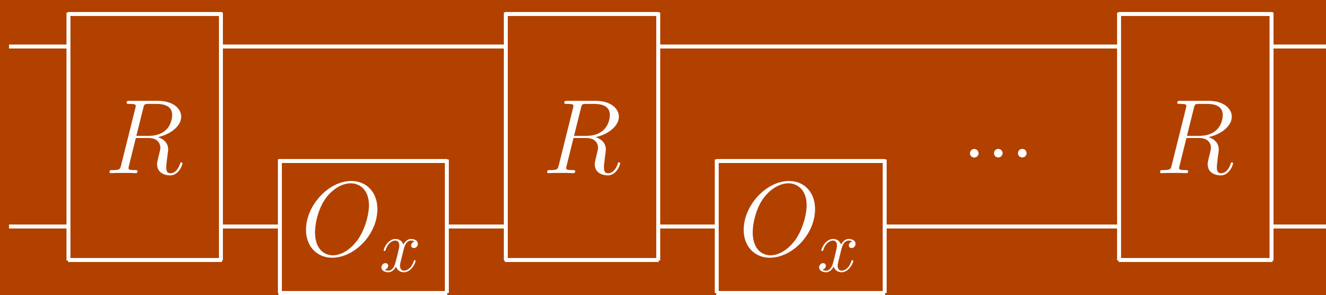
$$\parallel \quad \vec{v} \in \Delta^\perp$$

$\Rightarrow \Omega(1/W)$ effective spectral gap

$\Rightarrow$ Running phase estimation with precision 1/100W,

with prob. $\approx$1/2 measure eigenvalue 1, leaving $\frac{1}{\sqrt{2}}\big(|r\rangle + |f(x)\rangle\big)$!

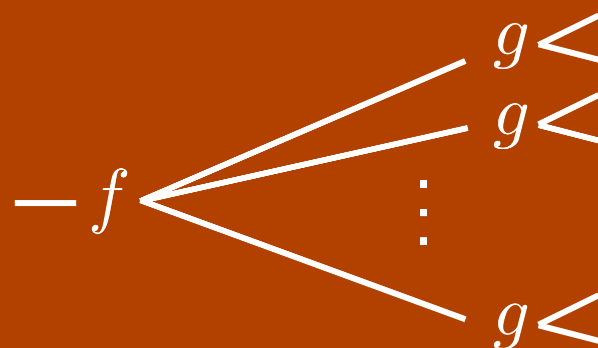
Corollaries



R
O_x
R
O_x
$\cdots$
R

How does query complexity change under composition?

Model: For $f : \{0,1\}^n \to \{0,1\},\ g : \{0,1\}^m \to \{0,1\}$
let $f \bullet g$ be the function $f \circ (g, g, \ldots, g)$



- Deterministic query complexity
  $D(f \bullet g) = D(f)D(g)$
- Randomized
  $R(f \bullet g) \leq R(f)R(g) \cdot O(\log n)$
- Certificate complexity
  $C(f \bullet g) \leq C(f)C(g)$
  (can be $\ll$)

Model: For $f : \{0,1\}^n \to \{0,1\},\ g : \{0,1\}^m \to \{0,1\}$
let $f \bullet g$ be the function $f \circ (g, g, \ldots, g)$

- Deterministic query complexity
  $D(f \bullet g) = D(f)D(g)$
- Certificate complexity
  $C(f \bullet g) \le C(f)C(g)$
  (can be $\ll$)
- Randomized
  $R(f \bullet g) \le R(f)R(g) \cdot O(\log n)$

- Quantum query complexity

$$\boxed{\text{Claim: } \mathrm{Adv}^{\pm}(f \bullet g) = \mathrm{Adv}^{\pm}(f)\mathrm{Adv}^{\pm}(g)}$$

$$\Rightarrow Q(f \bullet g) = \Theta\big(Q(f)Q(g)\big)$$

$$\Rightarrow Q(f_1 \bullet f_2 \bullet \cdots \bullet f_d) = \Theta\big(\mathrm{Adv}^{\pm}(f_1) \cdots \mathrm{Adv}^{\pm}(f_d)\big)$$

Proof idea for ≤ direction:

Let u, v and μ, ν be vector $\mathrm{Adv}^{\pm}$ solutions for f and for g

Then "u⊗μ", "v⊗ν" is a solution for f • g if domain $D \subseteq \{0,1\}^n$:

$$|\tilde{u}_x\rangle = \sum_j |j, x_j\rangle \otimes |u_{xj}\rangle$$

$$|\tilde{v}_y\rangle = \sum_j |j, \bar{y}_j\rangle \otimes |v_{xj}\rangle$$

$$\Rightarrow \quad \langle \tilde{u}_x | \tilde{v}_y \rangle = \sum_{j : x_j \neq y_j} \langle u_{xj} | v_{xj} \rangle$$

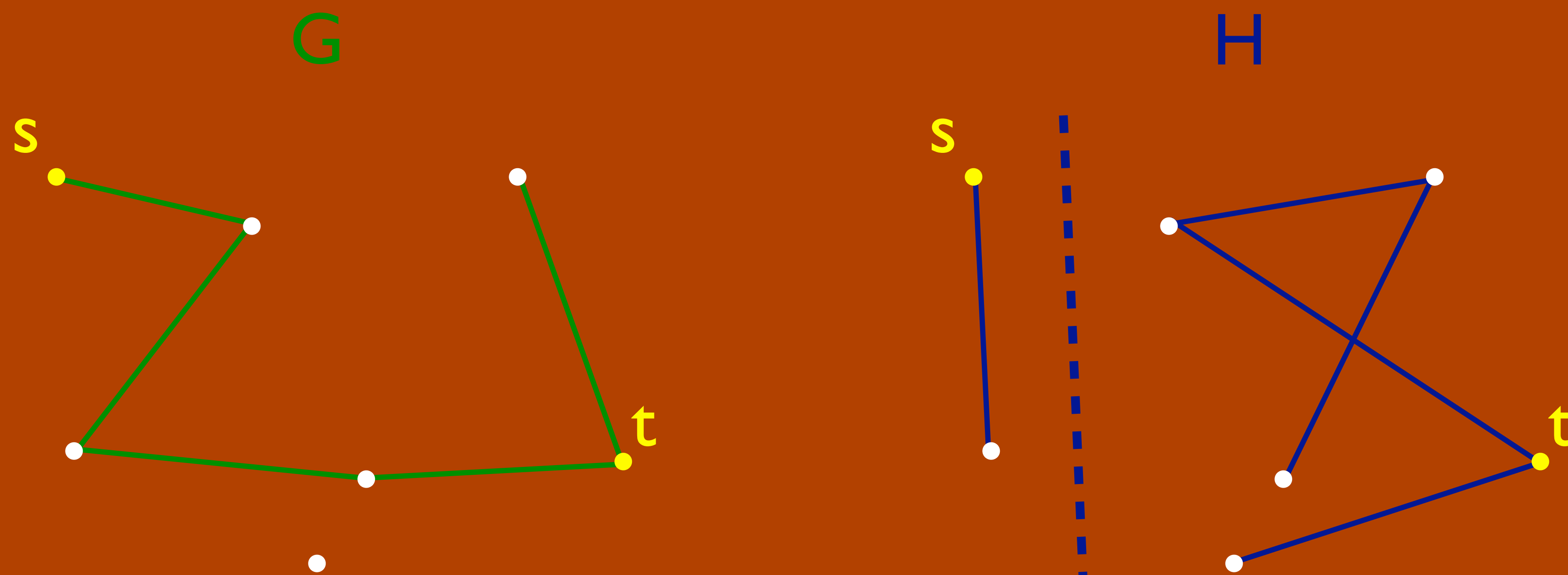
if $D \subseteq \{1,2,\ldots,k\}^n$:

let $|\mu_b\rangle = |0\rangle + |b\rangle, \ |\nu_b\rangle = |0\rangle - |b\rangle$

so $\langle \mu_b | \nu_{b'} \rangle = 1 - \delta_{b,b'}$

$$|\tilde{u}_x\rangle = \sum_j |j, \mu_{x_j}\rangle \otimes |u_{xj}\rangle \qquad |\tilde{v}_y\rangle = \sum_j |j, \nu_{y_j}\rangle \otimes |v_{yj}\rangle$$

Time complexity:
s-t Connectivity

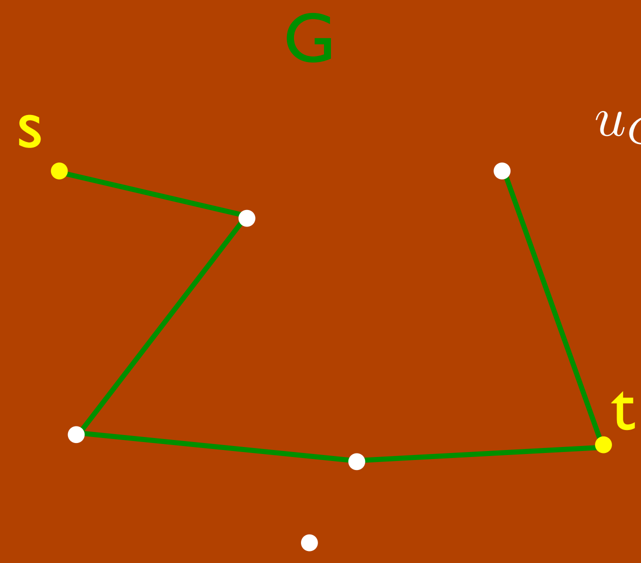


$$u_{Ge} = \begin{cases} 1 & \text{path crosses } e \text{ to the right} \\ -1 & \text{path crosses } e \text{ to the left} \end{cases}$$

$u_{He} = 1$ across the cut

$$\Rightarrow \sum_{e: e \in G, e \notin H} \langle u_{Ge} | u_{He} \rangle = (\#\text{ right crossings}) - (\#\text{ left crossings}) = 1$$

$$\Rightarrow \mathrm{Adv}^{\pm}(f) \leq n^{3/2}$$

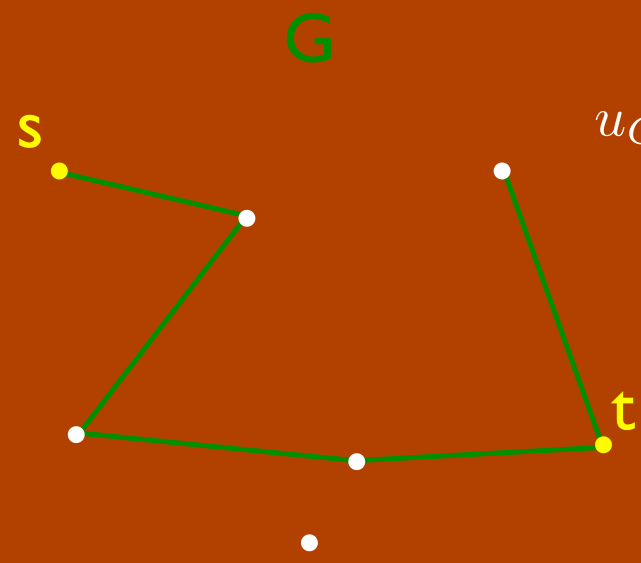


$$u_{Ge} = \begin{cases} 1 & e \in \text{path} \\ -1 & -e \in \text{path} \end{cases}$$

$$\Delta = \text{projection onto span}\{\psi_x\}$$

$$|\psi_x\rangle = |r\rangle + |f(x)\rangle + \frac{1}{10\sqrt{W}}|\tilde{u}_x\rangle$$

$$|\tilde{u}_x\rangle = \sum_j |j, x_j\rangle \otimes |u_{xj}\rangle$$

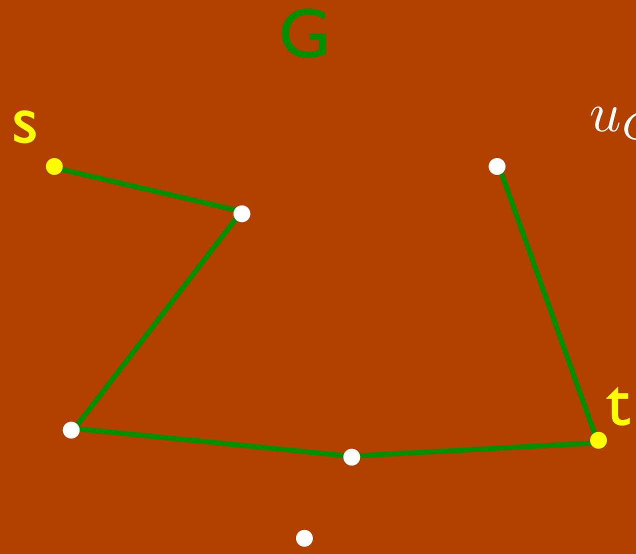


$$u_{Ge} = \begin{cases} 1 & e \in \text{path} \\ -1 & -e \in \text{path} \end{cases}$$

$$\Delta = \text{projection onto span}\{\psi_G\}$$

$$|\psi_G\rangle = |r\rangle + |f(G)\rangle + \frac{1}{10\sqrt{W}}|\tilde{u}_G\rangle$$

$$|\tilde{u}_G\rangle = \sum_e |e, G_e\rangle \otimes |u_{Ge}\rangle$$



$$u_{Ge} = \begin{cases} 1 & e \in \text{path} \\ -1 & -e \in \text{path} \end{cases}$$

$$\Delta = \text{projection onto span}\{\psi_G : f(G) = 1\}$$

$$|\psi_G\rangle = \cancel{|r\rangle + |f(G)\rangle} + \frac{1}{10\sqrt{W}}|\tilde{u}_G\rangle$$

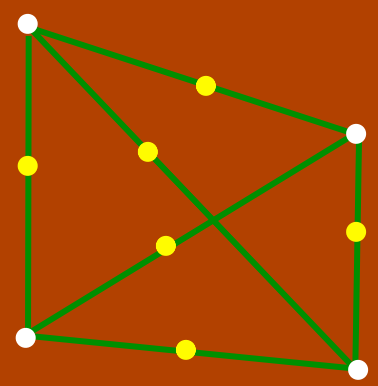
$$\cancel{|\tilde{u}_G\rangle = \sum_e |e, G_e\rangle \otimes |u_{Ge}\rangle}$$

$$|\tilde{u}_G\rangle = \sum_e u_{Ge}|e\rangle$$

$$\Rightarrow \Delta \sim \text{proj. onto balanced } s\text{-}t \text{ flows in } K_n$$

Implementing the reflection about the set of balanced flows in K_n

Flow = $|\psi\rangle \in \mathbb{C}^E$ s.t. $\forall$v, $|\psi\rangle \perp \sum_w |(v,w)\rangle$

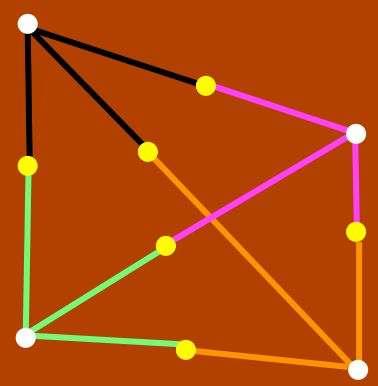


1. Factor the solution

- into constraints on original vertices & on edge vertices—now commuting

Implementing the reflection about the set of balanced flows in K_n

Flow = $|\psi\rangle \in \mathbb{C}^E$ s.t. $\forall$v, $|\psi\rangle \perp \sum_w |(v,w)\rangle$

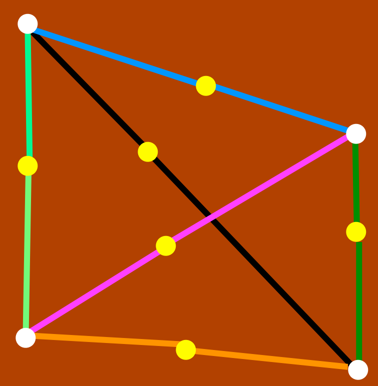


1. Factor the solution

- into constraints on original vertices & on edge vertices—now commuting

Implementing the reflection about the set of balanced flows in K_n

Flow = $|\psi\rangle \in \mathbb{C}^E$ s.t. $\forall$v, $|\psi\rangle \perp \sum_w |(v,w)\rangle$



1. Factor the solution

   - into constraints on original vertices & on edge vertices—now commuting

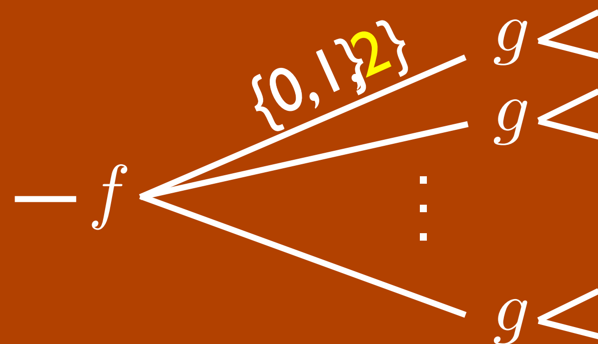
2. Apply Jordan's lemma (Szegedy)

3. Use phase estimation to isolate the +1 eigenspace, reflect, uncompute

Open problems

- More quantum algorithms

- Composition lower bounds with a non-boolean intermediate space



e.g., if f only depends on the input parities & g outputs even numbers

$$Q(f \bullet g) = 0 \ll Q(f)Q(g)$$

- Strong direct-product thms. for evaluating multiple indep. functions?

$$Q\big((g, g, \ldots, g)\big) = Q(\mathbf{1}_n \bullet g) = \Theta(nQ(g))$$

- Largest possible classical/quantum gap on total functions or sufficiently symmetrical functions

- Other query questions:

Evaluating relations

State generation

Query complexity with a bounded-error oracle

References

- Lower bound survey: Høyer & Špalek 0509153
- Adversary bound
  - Bennett, Bernstein, Brassard, Vazirani 9701001
  - Ambainis ’00
  - Høyer, Neerbek, Shi ’02
  - Ambainis 0305028
  - Barnum, Saks & Szegedy ’03
  - Laplante & Magniez 0311189
  - Zhang 0311060
  - Špalek & Szegedy 0409116
  - Barnum, Saks ’04
  - Multiplicative adversary: Ambainis 0508200; Ambainis, Špalek, de Wolf ’06; Špalek 0703237; Ambainis, Magnin, Roetteler, Roland 1012.2112
  - General adversary bound: Høyer, Lee, Špalek 0611054
- Today’s proofs: Lee, Mittal, Reichardt, Špalek ’10?

Blank slide