When Can a Polygon Fold to a Polytope?

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Abstract

We show that the decision question posed in the title can be answered with an algorithm of time and space complexity $O(n^2)$, for a polygon of n vertices. We use a theorem of Aleksandrov that says that if the edges of the polygon can be matched in length so that the resulting complex is homeomorphic to a sphere, and such that the "complete angle" at each vertex is no more than 2π, then the implied folding corresponds to a unique convex polytope. We check the Aleksandrov conditions via dynamic programming. The algorithm has been implemented and tested.

1 Introduction

The polygon shown in Fig. 1a cannot fold edge-to-edge to a convex polytope despite being composed of six squares that one might think could fold to a cube. The familiar "cross" polygon in Fig. 1b can of course fold to a cube. The aim of this paper is to provide an algorithm that can decide whether a polygon can fold to a polytope.

We take a *polygon* P to be a collection of vertices $(v_0, v_1, ..., v_{n-1})$ in the plane, connected by edges $(e_0, e_1, ..., e_{n-1})$ with $e_i = v_i v_{i+1}$. We do not insist that our polygons be simple. We only need the edges to be oriented consistently so that the interior angle α_i at vertex v_i is well-defined and less than 2π. We chose an orientation so that all indices increase in counterclockwise order. All index arithmetic is mod n. The polygonal chain from v_i counterclockwise to v_j is denoted by $P[i, j]$. A *polytope* $\mathcal{P}$ is a convex polyhedron that is the convex hull of a finite set of points in 3-space.

We assume a polygon is given by its edge lengths $\ell_i = |e_i|$ and interior angles α_i (which can be computed from vertex coordinates). We prefer not to concern ourselves with the bit complexity of the input, and will measure the input size with n, the number of lengths and angles input. Our question is: When can a polygon fold to a polytope? Equivalently, could

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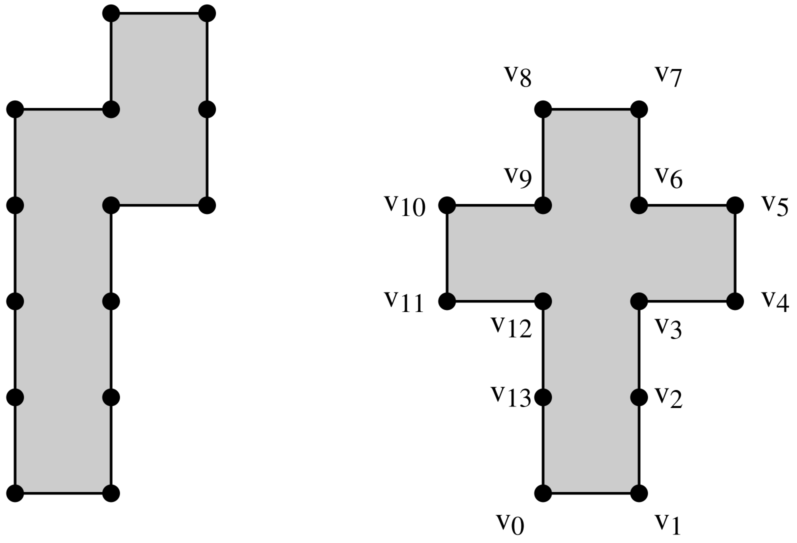

(a) (b)

Figure 1: (a) A polygon that cannot fold to a cube; (b) An unfolding of a cube.

some polytope surface be sliced and unfolded to produce the given polygon? We provide an $O(n^2)$ algorithm to answer this decision question.

We start by ruling out certain types of degeneracies: we insist that each vertex of the polygon corresponds to a strictly convex vertex of the polytope, and that every vertex of the polytope corresponds to some vertices of the polygon joined together. Thus vertex v_2 in Fig. 1b must be present even though the interior angle $\alpha_2 = \pi$. These conditions disallow vertices of the polygon folding to a point interior to a face of $\mathcal{P}$. Such "flat foldings" will be examined in Section 5.

2 Aleksandrov's Theorem

Cauchy proved a fundamental theorem over a century ago: if two polytopes have the same combinatorial structure, and the corresponding faces are congruent, then the polytopes are congruent [Whi94, p. 147]. In our terms, Cauchy's theorem requires knowing the fold (crease) lines. Aleksandrov erased this dependence with a powerful extension of Cauchy's theorem [Ale58]. One formulation of his result is as follows [Bus58, Thm. 17.1, p. 128]:

Theorem 1 *"A polyhedral metric with non-negative curvature on the sphere can be realized as one, and up to motions only one, (possibly degenerate) polyhedron."*

A *polyhedral metric* [AZ67, p. 8] assigns to each point a neighborhood that is either isometric to an open disk in E^2, or to the apex of a cone whose complete angle is less than 2π. The *complete angle* [Pog49, p. 15] of a vertex of a polytope is the sum of the incident face angles; the complete angle at the apex of a cone is the natural analog.

One way to translate this theorem to a form more useful for our purposes is via Aleksandrov's concept of a *net*: a collection of polygons whose edges are matched or identified so that:

1. Each edge is matched with one of equal length.
2. The sum of the angles incident to each vertex is no more than 2π.
3. The resulting complex is homeomorphic to a sphere.

Any such net defines a polyhedral metric, and therefore corresponds to a unique polytope (unique up to congruence). In our context his net is our single polygon. His theorem says that we only need check the obvious local necessary conditions for folding: matching edge lengths and achieving convex vertex angles. If these conditions hold, the polygon folds to a unique polytope according to that matching. Although a "legal" matching (one producing a net) results in a unique polytope, it is possible that there can be more than one legal matching. Indeed, Shephard found a polygon with two distinct matchings, which fold to two non-congruent polytopes [She75]. We will see examples of this phenomena in Section 5.

Aleksandrov's theorem does not require that the edges of the net/polygon fold to edges of the polytope. To reverse the viewpoint: the slicing that produces the polygon can cut through face interiors. Initially we will insist that the slicing start and terminate at vertices (but need not follow polytope edges), although his theorem covers a more general situation.

It should be admitted that for "generic" polytope unfoldings, the question we have posed is easy to answer. Consider the polygon in Fig. 2. It was produced by unfolding the convex hull of random points in a sphere. The random generation produced distinct polytope edge lengths. Thus each edge of the polygon has only one possible mate: the only other one with the same length. Checking Aleksandrov's conditions then amounts to checking the angle sum at the vertices of the unique gluing. The problem is more difficult when many edges have the same length, and so there are many choices for edge matches. In the examples in Fig. 1 (and Fig. 4 below), all edge lengths are identical. Naive search would explore an exponential number of possible matches.

3 Dynamic Programming Formulation

The basic insight of our algorithm is that if e_i is matched with e_j, which we write as $e_i{\equiv}e_j$, then we have two smaller problems: folding $P[i+1,j]$ and $P[j+1,i]$. These two subproblems are isolated from one another by the two matched edges. For if an edge in $P[i+1,j]$ is glued to an edge in $P[j+1,i]$, any resulting surface will not be homeomorphic to a sphere. Certain choices of initial edge matches lead to subproblems with no solution. Others lead to a subproblem which can be solved but which is incompatible with the angle conditions at the endpoints of e_i and e_j. And others lead to compatible, solvable subproblems, which result in a folding. We design an algorithm based on dynamic programming, formulated as follows.

We write $v_i{\equiv}v_j$ to mean that these two vertices are identified. The key quantity associated with the matching $v_i{\equiv}v_j$ is $\alpha_{min}(i,j)$. We first provide an informal definition. Define $\alpha_{min}(i,j)=\infty$ if there is no "legal" folding of the chain $P[i,j]$. If there is at least one legal

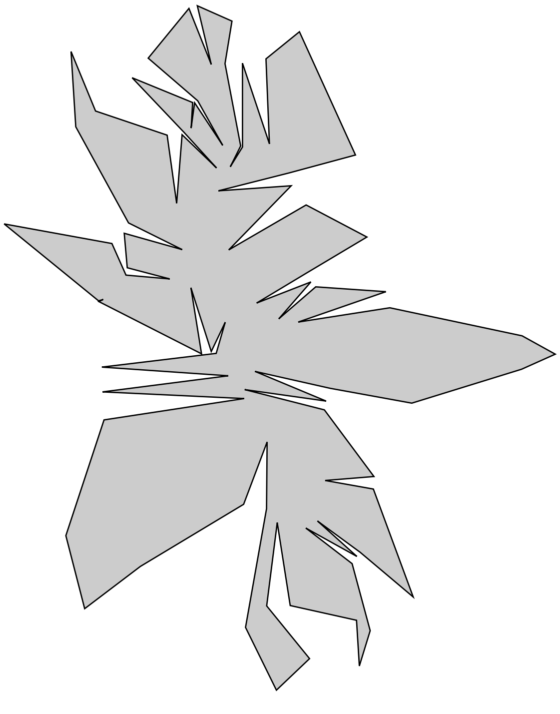

Figure 2: An unfolding of a randomly-generated convex polytope. The polytope was generated as in [O'R94, p. 155] and the unfolding produced according to [NF93].

folding of the chain, $\alpha_{min}(i,j)$ is the minimum face angle incident to $v_i = v_j$ contributed by the chain, minimum over all legal foldings of $P[i,j]$. A formal definition is as follows.

D1. For $|j-i|$ odd, $\alpha_{min}(i,j) = \infty$.

D2. For $i = j$, $\alpha_{min}(i,j) = 0$.

D3. For $|j-i| \geq 2$ and even, let k have different parity from i, and $i < k \leq j-1$. $\alpha_{min}(i,j)$ is the minimum over all such k of the "extra" angle ϵ_k at $v_i = v_j$ resulting from a folding that matches $e_i{\equiv}e_k$. If $\ell_i \neq \ell_k$, then $\epsilon_k = \infty$. Otherwise this match creates two subproblems (see Fig. 3):

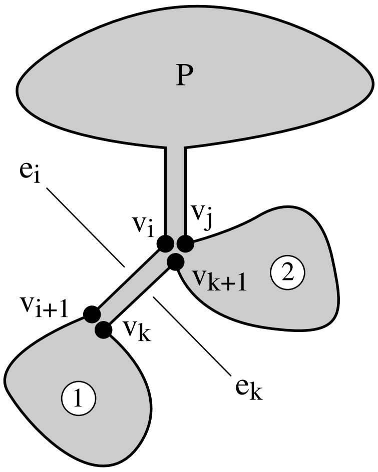

Figure 3: The match $v_i{\equiv}v_j$ creates two subproblems for every possible gluing $e_i{\equiv}e_k$.

1. $v_{i+1}{\equiv}v_k$. If $i+1 = k$, this subproblem is vacuous and ϵ_k is determined by the second subproblem. Assume then that $i+1 \neq k$. If $\alpha_{i+1} + \alpha_k + \alpha_{min}(i+1,k) \geq 2\pi$, the gluing $e_i{\equiv}e_k$ is not possible, so $\epsilon_k = \infty$. Otherwise ϵ_k is determined by the second subproblem:
2. $v_{k+1}{\equiv}v_j$. The extra angle is 0 if $k+1 = j$, and $\epsilon_k = \alpha_{k+1} + \alpha_{min}(k+1,j)$ otherwise.

D1 of this definition simply reflects the fact that between any two matched vertices must lie an even number of edges, because the edges are matched in pairs. D2 says that a vertex can always match with itself (which happens when it is the endpoint of a slice on the polytope surface), resulting in no extra angle glued to $v_i = v_j$.

The heart of the procedure is D3, which we illustrate with the cube unfolding in Fig. 1b, computing $\alpha_{min}(1,5)$, corresponding to matching $v_1{\equiv}v_5$. There are two values of k within the relevant range: $k = 2$ and $k = 4$. The first corresponds to matching $e_1{\equiv}e_2$. This creates two subproblems, $v_2{\equiv}v_2$ and $v_3{\equiv}v_5$. The first is possible (D1) and so we turn to the second, which yields an extra angle of $\epsilon_2 = \alpha_3 + \alpha_{min}(3,5) = 270° + 0 = 270°$. The $k = 4$ case corresponds to the match $e_1 \equiv e_4$, which creates the two subproblems $v_2 \equiv v_4$ and $v_5 \equiv v_5$. The first is legal since $\alpha_2 + \alpha_4 + \alpha_{min}(2,4) = 180° + 90° + 0 = 270° < 360°$. The second yields $\epsilon_4 = 0$ (by D1). Finally, $\alpha_{min}(1,5) = \min\{\epsilon_2, \epsilon_4\} = 0$, meaning that there is a way to fold the chain $P[1,5]$ so that no extra angle is matched to $v_1 = v_5$.

It is clear that $\alpha_{min}(i,j)$ can be computed for all i and j by dynamic programming: values for $|j-i| = d$ depend on values of α_{min} for strictly smaller index separation. Thus the entire table of α_{min} values can be computed in the order: $d = 0, 2, 4, ..., n-2$.

With this table in hand, the question of whether or not P can fold may be answered by seeing if e_0 can be glued to some other edge e_m. This is answered by examining the two subproblems exactly as in D3 of the α_{min} definition[1] $v_1 \equiv v_m$ and $v_{m+1} \equiv v_0$, and then verifying that the angle conditions are satisfied at the endpoints of $e_0 = e_m$:

$$\begin{aligned} \alpha_1 + \alpha_m + \alpha_{min}(1,m) &< 2\pi \\ \alpha_{m+1} + \alpha_0 + \alpha_{min}(m+1,0) &< 2\pi \end{aligned}$$

It should be clear that the entire algorithm takes $O(n^2)$ time, $O(n)$ time for all α_{min} entries separated by $|j-i| = d$. Storing the table requires quadratic space.

4 Octahedron Example

We will show the complete dynamic programming table for one example. Consider the polygon in Fig. 4, which is an unfolding of an octahedron. It has $n = 10$ vertices, whose angles (in degrees) are as follows:

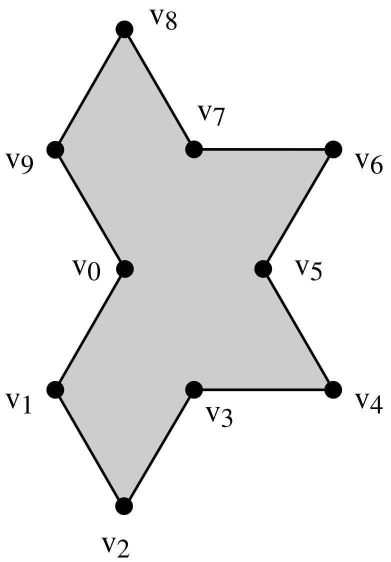

Figure 4: An unfolding of an octahedron.

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[1] In fact one could extend the table to $d = n$ to capture this case.

i	0	1	2	3	4	5	6	7	8	9
α_i	240	120	60	240	60	240	60	240	60	120

All edge lengths are identical, $\ell_i = 1$. After initializing $\alpha_{min}(i,j) = 0$ for $|j-i| \in \{0,2\}$, we compute the table for $|j-i| = 4$ using D3:

(i,j)	(0,4)	(1,5)	(2,6)	(3,7)	(4,8)	(5,9)	(6,0)	(7,1)	(8,2)	(9,3)
$\alpha_{min}(i,j)$	60	0	60	0	60	0	60	0	0	0

We will explain the computation of the first entry of this table, $\alpha_{min}(0,4) = 60°$. D3 explores two values of k, $k = 1, 3$, corresponding to the two ways to fold the four edges in $P[0,4]$. The first glues $e_0 \equiv e_1$, which forces $e_2 \equiv e_3$. Both of these matches are legal foldings. This folding brings v_2 to match $v_0 = v_4$, and so the extra angle is $\epsilon_1 = \alpha_2 = 60°$. The second way to fold the four edges is to glue $e_0 \equiv e_3$ and $e_1 \equiv e_2$. The first subproblem of D3 is $v_1 \equiv v_3$, which results in the angle $\alpha_1 + \alpha_3 + \alpha_{min}(1,3) = 120° + 240° + 0 = 360°$ which is not a legal folding, as the angle is not strictly less than 2π. Thus $\epsilon_3 = \infty$. Finally, $\alpha_{min}(0,4) = \min\{\epsilon_1, \epsilon_3\} = 60°$.

The tables for $|j-i| \in \{6,8\}$ follow:

(i,j)	(0,6)	(1,7)	(2,8)	(3,9)	(4,0)	(5,1)	(6,2)	(7,3)	(8,4)	(9,5)
$\alpha_{min}(i,j)$	120	0	120	0	120	120	60	0	60	120

(i,j)	(0,8)	(1,9)	(2,0)	(3,1)	(4,2)	(5,3)	(6,4)	(7,5)	(8,6)	(9,7)
$\alpha_{min}(i,j)$	180	0	180	120	120	0	120	0	120	120

Notice that no entry of any of the tables is ∞, indicating that all the implied subfoldings are legal. This is of course not always the case.

We come now to the final set of tests: finding a match for e_0. We will step through two matches, $m = 5$ and $m = 9$.

1. $m = 5$: $e_0 \equiv e_5$. The two subproblems are $v_1 \equiv v_5$ and $v_6 \equiv v_0$. The first results in this angle calculation:

   $$\alpha_1 + \alpha_5 + \alpha_{min}(1,5) = 120° + 240° + 0 = 360°$$

   And thus e_0 cannot be glued to e_5.

2. $m = 9$: $e_0 \equiv e_9$. The two subproblems are $v_1 \equiv v_9$ and $v_0 \equiv v_0$. The second is trivially legal; the first induces this angle calculation:

   $$\alpha_1 + \alpha_9 + \alpha_{min}(1,9) = 120° + 120° + 0 = 240°$$

   This represents the only legal folding of the polygon, whose complete set of edge matchings is:

   $$\{e_0 \equiv e_9 \, , e_1 \equiv e_8 \, , e_2 \equiv e_3 \, , e_4 \equiv e_5 \, , e_6 \equiv e_7\}.$$

5 Flat Folding

We have interpreted the angle condition to demand strictly less than 2π at each vertex. Aleksandrov's theorem holds when the complete angle at each vertex is no more than 2π. Permitting equality with 2π can be interpreted as follows: every true (strictly convex) vertex of the polytope $\mathcal{P}$ maps to one or more vertices of the polygon P, but some vertices of the polygon may not correspond to any vertex of the polytope: rather they "fold flat" to interior points of a polytope face or edge. We illustrate with two examples, the first in detail, the second briefly.

Permitting flat foldings in the above sense leads to no less than five distinct ways to fold the "cross" unfolding of the cube shown in Fig. 1b, producing these five polytopes:

1. The cube.

2. A flat, doubly-covered quadrilateral, a degenerate polytope permitted by Aleksandrov's theorem:

$$\{e_0{\equiv}e_3\ , e_1{\equiv}e_2\ , e_4{\equiv}e_{13}\ , e_5{\equiv}e_6\ , e_7{\equiv}e_{12}\ , e_8{\equiv}e_{11}\ , e_9{\equiv}e_{10}\ \}.$$

   The vertices of the quadrilateral are $v_0 = v_4$, v_2, v_6, and v_{10}.

3. A tetrahedron:

$$\{e_0{\equiv}e_1\ , e_2{\equiv}e_3\ , e_4{\equiv}e_{13}\ , e_5{\equiv}e_6\ , e_7{\equiv}e_{12}\ , e_8{\equiv}e_{11}\ , e_9{\equiv}e_{10}\ \}.$$

   In this matching, only four vertices have a complete angle strictly less than 2π: v_1, v_3, v_6, and v_{10}. The folding to a tetrahedron is shown in Fig. 5. Note, for example, that the set of vertices $\{v_5, v_7, v_{13}\}$ join in the interior of one of the tetrahedron's faces: $\alpha_5 + \alpha_7 + \alpha_{13} = 90° + 90° + 180° = 360°$; and the pair of vertices $\{v_9, v_{11}\}$ join at the interior of an edge: $\alpha_9 + \alpha_{11} = 270° + 90° = 360°$;

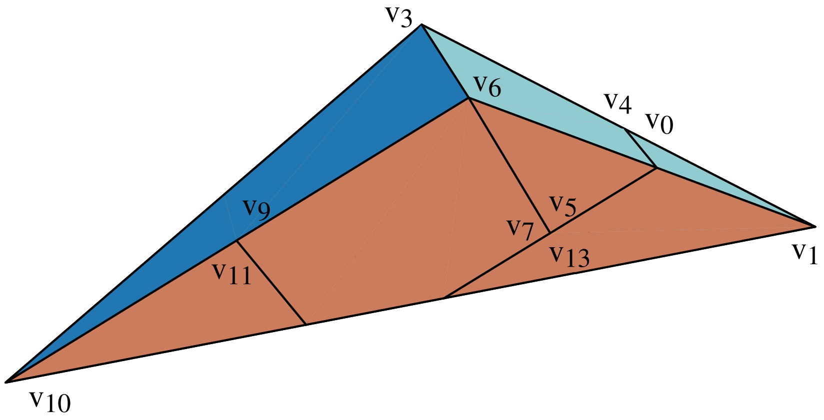

Figure 5: Refolding the cube unfolding of Fig. 1b to a tetrahedron.

4. A pentahedron:

$$\{e_0{\equiv}e_3\ ,e_1{\equiv}e_2\ ,e_4{\equiv}e_9\ ,e_5{\equiv}e_8\ ,e_6{\equiv}e_7\ ,e_{10}{\equiv}e_{13}\ ,e_{11}{\equiv}e_{12}\ \}.$$

5. An octahedron:

$$\{e_0{\equiv}e_3\ ,e_1{\equiv}e_2\ ,e_4{\equiv}e_7\ ,e_5{\equiv}e_6\ ,e_8{\equiv}e_9\ ,e_{10}{\equiv}e_{13}\ ,e_{11}{\equiv}e_{12}\ \}.$$

We have thus established the curious fact that a cube may be unfolded and then refolded to several distinct polytopes.

Similarly, the polygon of Fig. 1a can fold to several flat, doubly-covered polytopes: two different quadrilaterals, a pentagon, a hexagon; as well as to several nondegenerate polytopes: a pentahedron, a hexahedron, and an octahedron.

6 Future Work

It is natural to next ask for an algorithm to construct a three-dimensional representation of the unique polytope resulting from a folding. This seems difficult. Perhaps a more approachable problem is to remove the restriction that matches must glue whole edges: Aleksandrov's theorem countenances identifying arbitrary subparts of the polygon boundary. We delimited the choices by vertices only for the convenience of our algorithm.

Acknowledgements

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