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# Counting Bubbles in Linear Chord Diagrams

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**Abstract**

In a linear chord diagram, a short chord is one that joins adjacent vertices. We define a bubble to be a region in a linear chord diagram devoid of short chords. We derive a formal generating function counting bubbles by their size and find an exact result for the mean bubble size. We find that once one discards diagrams that have no short chords at all, the distribution of bubble sizes is given by a smooth function in the limit of long diagrams. Using a summation over short chords, we find the exact form of this asymptotic distribution.

## 1 Introduction and basic notions

A linear chord diagram consists of a linear arrangement of $2n$ vertices. Each vertex is joined to exactly one different vertex by an unoriented arc called a *chord*. Hence every linear chord diagram on $2n$ vertices has exactly $n$ chords. As the chords are distinguished only by the positions of their endpoints, it is evident that there are $(2n-1)(2n-3)\cdots 1 = (2n-1)!!$ different linear chord diagrams on $2n$ vertices.

One interesting way of refining this counting[1] is by the number of so-called *short* chords, i.e., chords that join adjacent vertices. Kreweras and Poupard [3] provided recurrence relations and closed form expressions for the number of diagrams with exactly $\ell$ short chords. They also showed that the mean number of short chords is 1, which implies that the total number of short chords is equinumerous with the total number of linear chord diagrams,

[1] The combinatorics of linear chord diagrams has a long history beginning with Touchard [7] and Riordan's [5] studies of the number of chord crossings; cf. Pilaud and Rué [4] for a modern approach and further developments. Krasko and Omelchenko [2] provide a more complete list of references.

cf. [1]. Kreweras and Poupard [3] showed further that all higher factorial moments of the distribution approach 1 in the $n \to \infty$ limit, thus establishing the Poisson nature of the asymptotic distribution.

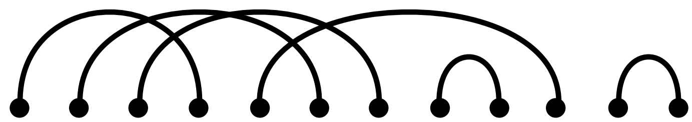

Figure 1: A linear chord diagram on 12 vertices consisting of one bubble of size 1 (bounded by the two short chords) and another bubble of size 7 (bounded by the start of the diagram and one of the short chords).

In this paper we will be concerned with counting certain sets of adjacent vertices of a linear chord diagram. We call these sets *bubbles*. The vertices of a given bubble may be joined by chords to one another or to vertices outside, but (in either case) never via a short chord. A bubble is therefore bounded either by short chords or by the ends of the diagram, see Figure 1. The *size* of a bubble is defined as the number of vertices it has, and may generically take values from 1 to $2n-2$, or, in the case of a linear chord diagram devoid of short chords, $2n$.

Let the total number of bubbles of size $p$ found among linear chord diagrams on $2n$ vertices be given by $B_{n,p}$. Table 1 shows the values of $B_{n,p}$ for $n \leq 6$.

| $n \backslash p$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | | | | | | | | | | |
| 2 | 2 | 0 | 0 | 1 | | | | | | | | |
| 3 | 8 | 4 | 2 | 2 | 0 | 5 | | | | | | |
| 4 | 42 | 30 | 20 | 15 | 12 | 10 | 0 | 36 | | | | |
| 5 | 300 | 240 | 186 | 147 | 120 | 99 | 82 | 72 | 0 | 329 | | |
| 6 | 2730 | 2310 | 1920 | 1605 | 1356 | 1155 | 988 | 848 | 730 | 658 | 0 | 3655 |

Table 1: Total number $B_{n,p}$ of bubbles of size $p$ counted across all possible linear chord diagrams with $n$ chords, On-line Encyclopedia of Integer Sequences A367000. The last entries in each row are given by A278990.

There are several interesting patterns to note in Table 1. First of all, we note that since a single diagram can consist of many bubbles, the row-sums are generically greater than the total number, $(2n-1)!!$, of linear chord diagrams. Secondly, the final entries in each row are the number of configurations consisting of a single bubble whose size is the length of the diagram; these are therefore linear chord diagrams devoid of short chords. These configurations have been studied elsewhere, for example by Kreweras and Poupard [3], and the On-line Encyclopedia of Integer Sequences entry A278990 gives these numbers as part of the greater sequence $d_{n,s}$ (A079267) of linear chord diagrams refined by the number $s$ of short chords. We may state therefore that $B_{n,2n} = d_{n,0}$. Thirdly, a bubble of size $2n-1$ can not be formed (as a short chord occupies two vertices), therefore the penultimate entry in each row is naturally zero: $B_{n,2n-1} = 0$. Fourthly, the third-from-last entry in each row is double the last entry of the row above. This is because a bubble of size $2n-2$ can be formed in only two ways: by placing a short chord at either end of the diagram, thus producing a linear chord diagram devoid of short chords on two fewer vertices. Hence $B_{n,2n-2} = 2B_{n-1,2n-2}$. One may try to continue this logic, for example, to the fourth-from-last entry. Bubbles of size $2n-3$ are formed by a short chord positioned one vertex away from either end of the diagram. There is therefore a chord that connects the first (or last) vertex of the diagram with a vertex within the bubble. We may therefore construct the bubble by starting with configurations devoid of short chords of length $2n-4$ and inserting this vertex in one of the $2n-3$ gaps between existing vertices, or start with configurations of length $2n-4$ containing a single short chord and place this vertex within it. We therefore have the following relation:

$$B_{n,2n-3} = 2\left((2n-3)B_{n-2,2n-4} + d_{n-2,1}\right).$$

There will be similar relations as we move further in to each row of the table; they will, however, become increasingly complex.

# 2 Enumeration of bubbles

## 2.1 Matching polynomial method

The present author [10] developed a technique for computing generating functions that count linear chord diagrams refined by short chords. As we will use this method in the following subsection to enumerate bubbles, we give an account of the method here first. The method centres upon the matching (or rook) polynomial for the path of length $2n$. We remind the reader that the matching polynomial $m_G(z) = \sum \rho_j z^j$ of a graph $G$ has coefficients $\rho_j$ that count the number of $j$-edge matchings on $G$. By convention the number $\rho_0$ of zero-edge matchings is defined to be 1 for every graph. For example, the path consisting of the four vertices $A$, $B$, $C$, and $D$, has three 1-edge matchings: $(AB)$, $(BC)$, and $(CD)$. It has one 2-edge matching: $(AB)(CD)$. The matching polynomial for the path of length four is therefore $1 + 3z + z^2$.

The number $d_{n,0}$ of linear chord diagrams on $2n$ vertices devoid of short chords may be calculated from the $\rho_j$ associated with the path of length $2n$ via inclusion-exclusion:

$$d_{n,0} = \sum_{j=0}^{n} (-1)^j (2n-2j-1)!!\, \rho_j. \tag{1}$$

The explanation is as follows. We note that for each of the $\rho_j$ choices of $j$ edges on which to place $j$ short chords, there remains $(2n-2j-1)!!$ configurations on the remaining $2n-2j$

vertices. There will be some number of configurations with exactly $q$ short chords among these $(2n-2j-1)!!$. Then $(2n-2j-1)!!\,\rho_j$ counts the number $d_{n,q+j}$ of $(q+j)$-short-chord configurations $\binom{q+j}{j}$ times. We thus have

$$\sum_{j=0}^{n}(-1)^j(2n-2j-1)!!\,\rho_j = \sum_{j=0}^{n}(-1)^j\sum_{q=0}^{n-j}\binom{q+j}{j}d_{n,q+j}$$

$$= d_{n,0} + \sum_{q+j=1}^{n} d_{n,q+j}\sum_{j=0}^{q+j}(-1)^j\binom{q+j}{j},$$

and so all but the 0-short-chord configurations cancel.

The alternating sum in Equation (1) may be repackaged as an integral involving the matching polynomial. It is more convenient to collect the matching polynomials for all paths together into a two-variable generating function, cf. [10, Proposition 15]:

$$L(x,y) = \frac{1}{1-y(1+x^2y)},$$

where $[y^{2n}]L(x,y)$ is the matching polynomial $m(x^2)$ for the path consisting of $2n$ vertices; the replacement $z \to x^2$ is a useful redefinition for the calculations that follow.

The generating function for the $d_{n,0}$ is then given by

$$\sum_n d_{n,0}\, y^{2n} = \int_0^\infty dt\, e^{-t}\,\frac{1}{2\pi i}\oint_{|x|=\epsilon}\frac{dx}{x}\,e^{x^2/2}\,L\left(ix/t, yt/x\right).$$

The explanation of this equality is as follows. A generic term in the expansion of $L\left(ix/t, yt/x\right)$ will have the form

$$\left(-\frac{x^2}{t^2}\right)^j\left(\frac{y^2t^2}{x^2}\right)^n\rho_j,$$

where odd powers of $y$, though present in $L(x,y)$, do not survive the contour integration over $x$, and have thus been omitted. The only term in the expansion of the exponential $\exp(x^2/2)$ surviving the contour integration will be $x^{2(n-j)}/(2^{n-j}(n-j)!)$, as this will absorb the $j-n$ powers of $x^2$ in the expansion of $L\left(ix/t, yt/x\right)$. Finally, the integration over $t$ provides a factor of $(2n-2j)!$. We therefore have that

$$\int_0^\infty dt\, e^{-t}\,\frac{1}{2\pi i}\oint_{|x|=\epsilon}\frac{dx}{x}\,e^{x^2/2}\,L\left(ix/t, yt/x\right) = \sum_{j=0}^{n}(-1)^j\frac{(2n-2j)!}{2^{n-j}(n-j)!}\,\rho_j,$$

which is Equation (1).

## 2.2 Application of the method to the enumeration of bubbles

A bubble is bounded to the left and right by short chords (or the ends of the diagram), thus we define the following generating function:

$$\mathcal{L}(r,w,x,y) = \frac{1}{1-y^2L(x,ry)}\Big(L(x,wry)-1\Big)\frac{1}{1-y^2L(x,ry)}.$$

The central factor $L(x,wry)-1$ corresponds to the bubble being counted, whereas the two factors of $(1-y^2L(x,ry))^{-1}$ correspond to the remainder of the diagram, to the left and to the right of the bubble; the presence of $y^2$ corresponds to the short chords bounding the bubble. In the expansion of $\mathcal{L}(r,w,x,y)$, each power of $y$ corresponds to a vertex of the linear chord diagram. Those vertices belonging to the bubble under consideration are further labelled with a power of $w$, while every vertex not part of a short chord is also labelled with a power of $r$. Armed with this aggregate generating function $\mathcal{L}(r,w,x,y)$ for the matching polynomial, we calculate the generating function that counts bubbles as follows. Let $\rho_j$ be the number of $j$-edge matchings on the vertices marked with $r$ and let $2q$ be the number of these vertices, then

$$\sum_{j=0}^{q}(-1)^j\,(2(q-j)-1)!!\,\rho_j = \sum_{j=0}^{q}(-1)^j\frac{(2(q-j))!}{2^{q-j}(q-j)!}\,\rho_j \tag{2}$$

counts the number of configurations with no short chords on the $2q$ vertices in question.

**Theorem 1.** *The generating function that counts the numbers $B_{n,p}$ is given by*

$$B(y^2,w) = \sum_{n,p} B_{n,p}\,y^{2n}w^p = \int_0^\infty dt\,e^{-t}\,\frac{1}{2\pi i}\oint_{|x|=\epsilon}\frac{dx}{x}\,e^{x^2/2}\,\mathcal{L}\left(t/x,w,ix/t,y\right).$$

*Proof.* A generic term in the expansion of $\mathcal{L}\left(t/x,w,ix/t,y\right)$ (which survives the contour integration) will have the form

$$\left(\frac{t^2}{x^2}\right)^q w^p\left(-\frac{x^2}{t^2}\right)^j y^{2n}\rho_j,$$

where $\rho_j$ are the aforementioned matching numbers. The third factor follows from the fact that $L(x,y)$ is an even function of $x$. The expansion of the exponential in $x^2$ will contribute only the term of order $x^{2(q-j)}$, as the contour integration in $x$ will eliminate all other terms. This mechanism also forces the power of $r$ to be even, and accounts for the even power of $t/x$ in the first factor. We note that

$$[x^{2(q-j)}]e^{x^2/2} = \frac{1}{2^{q-j}(q-j)!},$$

and that

$$\int_0^\infty dt\,e^{-t}\,\frac{(t^2)^q}{(-t^2)^j} = (-1)^j\,(2(q-j))!,$$

and so together they give the factor $(-1)^j\,(2(q-j)-1)!!$ as required by Equation (2). □

**Corollary 2.** *The generating function $B(y,w)$ is given by*

$$B(y,w) = \int_0^\infty dt\, e^{-t}\Bigg( \frac{(1-w)^2(1-wy)^2}{(1+w^2y)\,(1-w(1-wy))^2} \exp\frac{y}{2}\left(\frac{tw}{1+w^2y}\right)^2$$
$$+ \left( \frac{2-w(y+2)\,(1+y(1-w(2-wy)))}{(1-w(1-wy))^2} wy + \frac{1-wy}{1-w(1-wy)} t^2 wy^3 \right) \exp\frac{t^2y}{2} \Bigg).$$

*Proof.* This is established by evaluating the contour integral over $x$ from Theorem 1. We begin with a simplified expression for $\mathcal{L}\,(t/x, w, ix/t, y)$:

$$\mathcal{L}\,(t/x, w, ix/t, y) = \frac{wy(t-wxy)(x-ty+xy^2)^2}{(1+w^2y^2)} \frac{1}{(x-ty)^2\left(x - \frac{twy}{1+w^2y^2}\right)}.$$

We therefore have two poles: a simple pole at $x = twy(1+w^2y^2)^{-1}$, and a pole of order two at $x = ty$. Computing the residues at these poles we find, through direct calculation, the result for $B(y,w)$. □

**Lemma 3.** *The total number of bubbles, counted across all linear chord diagrams on $2n$ vertices is given by*

$$\sum_{p=1}^{2n} B_{n,p} = \frac{(2n-2)!(4n-5)}{2^{n-1}(n-1)!}.$$

*Proof.* We use Corollary 2 and set $w=1$:

$$B(y,1) = \sum_n y^n \sum_{p=1}^{2n} B_{n,p} = \int_0^\infty dt\, e^{-t}\left(1-y-(1-t^2)y^2 - t^2y^3\right)\exp\frac{t^2y}{2}$$
$$= \sum_j (2j-1)!!y^j\left(1-y-y^2+2j(2j+1)y^2(1-y)\right),$$

where in the second line we have used the expansion of the exponential. Our result is then obtained by reading off the coefficient of $y^n$:

$$[y^n]\sum_j (2j-1)!!y^j\left(1-y-y^2+2j(2j+1)y^2(1-y)\right) = \frac{(2n-2)!(4n-5)}{2^{n-1}(n-1)!}.$$

□

**Lemma 4.** *The un-normalized first moment of bubble size is given by*

$$\sum_{p=1}^{2n} p\, B_{n,p} = \frac{(2n-1)!}{2^{n-2}(n-2)!}.$$

*Proof.* We use Corollary 2 and take a derivative with respect to $w$:

$$\frac{\partial}{\partial w}B(y,1) = \sum_n y^n \sum_{p=1}^{2n} p\, B_{n,p} = \int_0^\infty dt\, e^{-t}\left(t^2 y(1-2y) - 2y\right)\exp\frac{t^2 y}{2}$$
$$= \sum_j (2j-1)!!y^j\left(-2y + 2j(2j+1)y(1-2y)\right),$$

where we have used the expansion of the exponential before integrating over $t$. Our result is then obtained by reading off the coefficient of $y^n$:

$$[y^n]\sum_j (2j-1)!!y^j\left(-2y + 2j(2j+1)y(1-2y)\right) = \frac{(2n-1)!}{2^{n-2}(n-2)!}.$$

□

**Theorem 5.** *The mean bubble size $\bar{p}$, taken over all bubbles on linear chord diagrams consisting of $2n$ vertices, is given by*

$$\bar{p} \equiv \frac{\sum_{p=1}^{2n} p\, B_{n,p}}{\sum_{p=1}^{2n} B_{n,p}} = \frac{2(2n-1)(n-1)}{4n-5}.$$

*Proof.* We employ the results of Lemmas 3 and 4. □

# 3 Asymptotic distribution

The values $B_{n,p}$ for $1 \leq p \leq 2n-2$ follow a smooth distribution in the limit $n \to \infty$, the form of which we shall discover in this section. It is therefore natural to discard the penultimate and final entries in Table 1, and to view the configurations consisting of a single bubble of size $2n$ as part of a different counting problem. Before we do this, however, we note that the result of Theorem 5 implies that the asymptotic value of the mean bubble size $\bar{p}$ is

$$\lim_{n\to\infty}\frac{2(2n-1)(n-1)}{4n-5} = n.$$

To see how this arises, we remind the reader that Kreweras and Poupard [3] proved that the number $k$ of short chords is asymptotically Poisson distributed with mean 1. To leading order, the presence of $k$ short chords induces $k+1$ bubbles. This implies

$$\sum_{p=1}^{2n} B_{n,p} \simeq (2n-1)!!\sum_{k\geq 0}(k+1)\frac{e^{-1}}{k!} = 2(2n-1)!!, \tag{3}$$

where $(2n-1)!!$ counts the total number of linear chord diagrams consisting of $n$ chords. Since the distribution of the positions of these short chords is asymptotically uniform, the mean size of a bubble is asymptotically $2n/(k+1)$. The overall mean bubble size is then

$$\frac{\sum_{p=1}^{2n} p\, B_{n,p}}{\sum_{p=1}^{2n} B_{n,p}} \simeq \frac{1}{2(2n-1)!!}(2n-1)!! \sum_{k\geq 0}(k+1)\frac{2n}{k+1}\frac{e^{-1}}{k!} = n. \tag{4}$$

We now turn our attention to the asymptotic distribution of bubble sizes, not including the values $p = 2n-1, 2n$. The mean of this distribution can be obtained by subtracting the contribution of the bubble of size $2n$ as follows:

$$\frac{\sum_{p=1}^{2n-2} p\, B_{n,p}}{\sum_{p=1}^{2n-2} B_{n,p}} \simeq \frac{\sum_{p=1}^{2n} p\, B_{n,p} - 2ne^{-1}(2n-1)!!}{\sum_{p=1}^{2n} B_{n,p} - e^{-1}(2n-1)!!} \simeq \frac{2-2e^{-1}}{2-e^{-1}}\, n \simeq 0.7746\, n, \tag{5}$$

where we have made use of Equation (3) and (4).

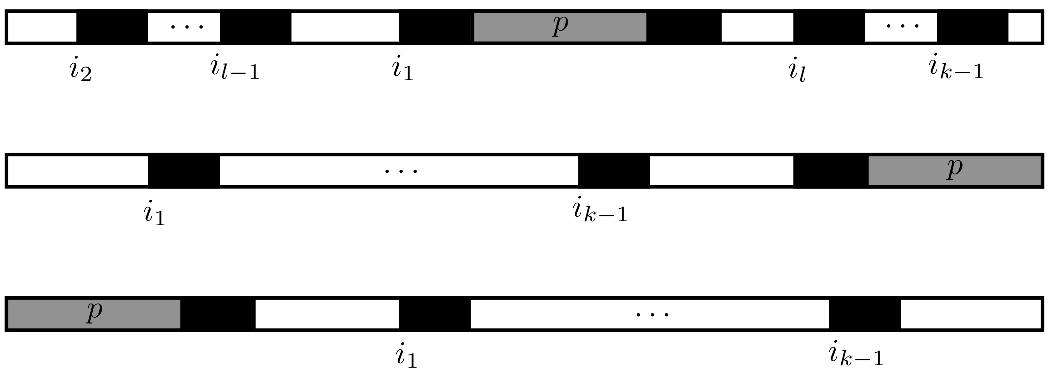


Figure 2: Contribution of diagrams with $k$ short chords (shown in black) to the asymptotic enumeration of bubbles of size $p$. The indices $i_j$ are positions that must be summed over.

**Theorem 6.** *Let $p$ denote the size of a bubble and let $x = p/(n-1)$. The asymptotic distribution $\rho(x)$ of the size of bubbles, excluding diagrams that are themselves bubbles, is given for $x \in (0, 2]$ by*

$$\rho(x) = \lim_{n\to\infty} \frac{(n-1)B_{n,x(n-1)}}{\sum_{p=1}^{2n-2} B_{n,p}} = \frac{e\,(6-x)}{2\,(4e-2)}\, e^{-x/2}.$$

*Proof.* We begin by noting that as $n$ is taken to infinity it is rare, amongst all linear chord diagrams, for a short chord to be found nested directly inside of another chord. Specifically, if $i-1, i, i+1, i+2$ are the positions of consecutive vertices, and $i, i+1$ are joined by a short chord, then it is unlikely that $i-1, i+2$ will also be joined by a chord. To see this consider the diagrams with exactly one short chord on $2n-2$ vertices. If one then inserts an additional short chord into one of these diagrams (to thus produce a diagram on $2n$ vertices), then there are $2n-1$ possible positions for it, only one of which will produce the nested configuration.

To leading order we may therefore consider a bubble to be constructed by the existence of some short chords (at least one). We then consider the vertices not participating in these short chords to constitute a (shorter) diagram devoid of short chords. The number $Z_n$ of diagrams on $2n$ vertices devoid of short chords is given asymptotically, according to the Poisson distribution, by $Z_n \simeq (2n-1)!!\, e^{-1}$, and thus $Z_n \simeq (2n-1)Z_{n-1}$.

Starting with a diagram devoid of short chords on $2(n-k)$ vertices, where $k$ is order 1, we add $k$ short chords to produce a diagram on $2n$ vertices. We may use one of these short chords (in conjunction with the end of the diagram), or two of them, to bound a bubble of size $p$ (which we take to be order $n$), see Figure 2. In the former case there are $k-1$ indistinguishable short chords whose positions must be summed over. In the latter, the complex consisting of the bubble and its two bounding chords must have its position summed over, in addition to the remaining $k-2$ indistinguishable chords. We therefore have the following asymptotic count for the number of bubbles of size $p$:

$$
\begin{aligned}
B_{n,p} &\simeq \sum_k Z_{n-k} \left( 2 \sum_{i_1 < \cdots < i_{k-1} = 1}^{2(n-k)-p-1} 1 + \sum_l \sum_{i_2 < \cdots < i_{l-1} < i_1 < i_l \cdots < i_{k-1} = 1}^{2(n-k)-p-1} 1 \right) \\
&\simeq \frac{Z_n}{2n-1} \sum_k \frac{1}{(2n - \mathcal{O}(1))^{k-1}} \left( 2 \frac{(2n-p-\mathcal{O}(1))^{k-1}}{(k-1)!} + \frac{(2n-p-\mathcal{O}(1))^{k-1}}{(k-2)!} \right) \\
&\simeq \frac{Z_n}{2n-1} \sum_k \frac{k+1}{2^{k-1}(k-1)!} (2-x)^{k-1},
\end{aligned}
$$

where we have used the relation $Z_n \simeq (2n-1)Z_{n-1}$ once, and then $k-1$ more times to express $Z_{n-k}$ in terms of $Z_n$. We recall from Equation (5) that

$$
\sum_{p=1}^{2n-2} B_{n,p} \simeq (2n-1)!! \left(2 - e^{-1}\right),
$$

and using $Z_n \simeq (2n-1)!!\, e^{-1}$, we therefore find

$$
\lim_{n\to\infty} \frac{(n-1)B_{n,x(n-1)}}{\sum_{p=1}^{2n-2} B_{n,p}} = \frac{1}{4e-2} \sum_k \frac{k+1}{2^{k-1}(k-1)!} (2-x)^{k-1} = \frac{e\,(6-x)}{2\,(4e-2)}\, e^{-x/2},
$$

where we have summed over $k$ from 1 to infinity. □

It is trivial to verify that the mean of $\rho(x)$ gives the result from Equation (5). In Figure 3 we have plotted the exact data for $n = 150$ against $\rho(x)$.

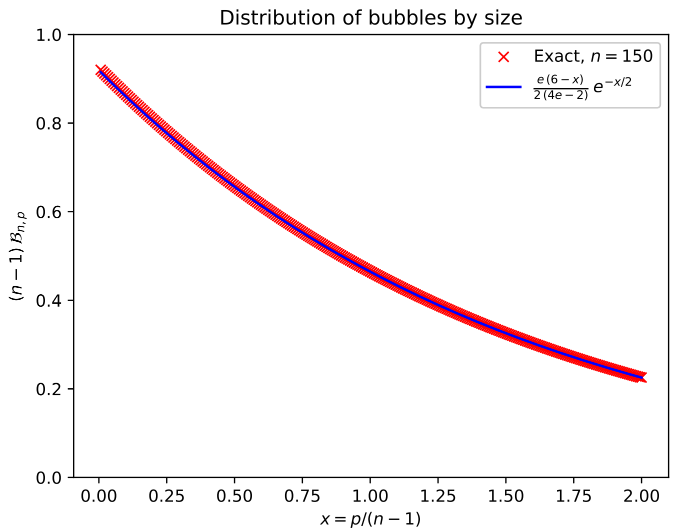


Figure 3: A plot displaying $(n-1)\mathcal{B}_{n,p} = (n-1)B_{n,p}/\sum_p B_{n,p}$ for $n = 150$ and for $p = 1$ to $2n-2$ in red “X”’s, and the asymptotic distribution $\rho(x)$ as a blue solid curve.

# 4 Open questions and further research

There are several directions in which the study of bubbles could be extended. An immediate question is whether the counts in Lemmas 3 and 4 can be argued more directly from combinatorial arguments. To give a sense of the difficulties present here, it is instructive to consider the total number of bubbles, i.e., the result of Lemma 3. One way of estimating the number of bubbles is to assume bubbles are bounded by single short chords (and the ends of the diagram). This means we are discounting configurations where short chords are adjacent to one another or found at the ends of the diagram. By counting the possible positions of a single short chord, and allowing the remaining $2n-2$ vertices to be connected in every possible manner, i.e.,

$$(2n-1)(2(n-1)-1)!!,$$

one is counting the diagrams with $s$ short chords $s$ times, and hence, according to the estimation, diagrams with $s+1$ bubbles $s$ times. We are thereby under-counting the number of bubbles by exactly the number of linear chord diagrams, i.e., $(2n-1)!!$. Adding these back we therefore find

$$\sum_{p=1}^{2n} B_{n,p} \simeq (2n-1)(2(n-1)-1)!! + (2n-1)!! = \frac{(2n-2)!(4n-2)}{2^{n-1}(n-1)!}.$$

This is a $1 + \mathcal{O}(1/n)$ multiplicative correction away from the exact result given in Lemma 3. The inclusion of the configurations where short chords are adjacent seems unwieldy, and the author has not found an elegant combinatorial argument to account for them.

Another direction of future research is to refine the counting of bubbles by the number of *internal* chords, i.e., chords with both their endpoints found within the bubble. The author [11] hopes to report his findings with respect to this problem shortly. Those chords that are not internal must bridge two bubbles; the connectivity of a given bubble to others in the same diagram becomes a natural further question.

Finally, the concept of a short chord extends readily to graphs other than the path of length $2n$, cf. Young [8][9], and it would be interesting to count bubbles on two (or higher) dimensional grids or other more general graphs.

## 5 Acknowledgment

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(Concerned with sequences A079267, A278990, and A367000.)

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