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On generalized Leonardo p-numbers

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Abstract

We introduce a generalization of Leonardo p-numbers, which extends the definition of Leonardo p-numbers given by Tan and Leung, and investigate some of their basic properties. Moreover, we give a companion matrix and derive some identities regarding the numbers.

1 Introduction

The sequence of the Fibonacci numbers A000045 $\{F_n\}_{n\geq 0}$ is the second-order linear recurrence sequence given by

$$F_0 = 0, F_1 = 1 \quad \text{and} \quad F_{n+1} = F_n + F_{n-1} \text{ for } n \geq 1.$$

One of the generalizations of Fibonacci numbers, given by Stakhov [7], is the Fibonacci p-numbers $F_p(n)$, which is defined for positive integers p by

$$F_p(n) = F_p(n-1) + F_p(n-p-1) \text{ for } n > p$$

with the initial values $F_p(0) = 0$ and $F_p(1) = F_p(2) = \cdots = F_p(p) = 1$.

Furthermore, Catarino and Borges in [2] introduced the Leonardo sequence $\{\mathrm{Le}_n\}_{n\geq 0}$ and derived some of its properties. It is the sequence A001595 satisfying the recurrence relation

$$\mathrm{Le}_0 = \mathrm{Le}_1 = 1 \quad \text{and} \quad \mathrm{Le}_n = \mathrm{Le}_{n-1} + \mathrm{Le}_{n-2} + 1 \text{ for } n \geq 2.$$

The first author and Chobsorn [4] presented the generalized Leonardo sequence and obtained some properties of this sequence. Recently, Tan and Leung [8] have introduced the sequence of the Leonardo p-numbers $\{\mathcal{L}_{p,n}\}_{n\geq 0}$ as

$$\mathcal{L}_p(n) = \mathcal{L}_p(n-1) + \mathcal{L}_p(n-p-1) + p \quad \text{for } n > p$$

with the initial values $\mathcal{L}_p(0) = \mathcal{L}_p(1) = \cdots = \mathcal{L}_p(p) = 1$. When $p = 1$, the Leonardo p-numbers reduce to the Leonardo numbers.

In this paper, we extend the aforementioned definition of the Leonardo p-numbers and derive some properties of the new sequence.

2 Generalized Leonardo p-numbers

We begin by defining a sequence of the generalized Leonardo p-numbers and establishing its basic properties.

Definition 1. For all fixed pairs of positive integers p and k, define the generalized Leonardo p-sequence $\{\mathcal{L}_{p,k}(n)\}_{n\geq 0}$ by

$$\mathcal{L}_{p,k}(n) = \mathcal{L}_{p,k}(n-1) + \mathcal{L}_{p,k}(n-p-1) + k \quad \text{for } n > p$$

with the initial values $\mathcal{L}_{p,k}(0) = \mathcal{L}_{p,k}(1) = \cdots = \mathcal{L}_{p,k}(p) = 1$.

Some special cases of the generalized Leonardo p-numbers are as follows.

- If $k = p$, the generalized Leonardo p-numbers are the Leonardo p-numbers.
- If $p = 1$, the generalized Leonardo p-numbers are the generalized Leonardo numbers.
- If $p = 1$ and $k = 2$, this sequence reduces to the sequence A111314.

We compute the first few terms of the sequences $F_2(n)$ and $\mathcal{L}_{2,k}(n)$ for $k = 1, 2, 3, 4, 5$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$F_2(n)$	1	1	1	2	3	4	6	9	13	19	28	41	60	88
$\mathcal{L}_{2,1}(n)$	1	1	3	5	7	11	17	25	37	55	81	119	175	257
$\mathcal{L}_{2,2}(n)$	1	1	4	7	10	16	25	37	55	82	121	178	262	385
$\mathcal{L}_{2,3}(n)$	1	1	5	9	13	21	33	49	73	109	161	237	349	513
$\mathcal{L}_{2,4}(n)$	1	1	6	11	16	26	41	61	91	136	201	296	436	641
$\mathcal{L}_{2,5}(n)$	1	1	7	13	19	31	49	73	109	163	241	355	523	769

Table 1: The first 14 terms of $F_2(n), \mathcal{L}_{2,1}(n), \mathcal{L}_{2,2}(n), \mathcal{L}_{2,3}(n), \mathcal{L}_{2,4}(n)$ and $\mathcal{L}_{2,5}(n)$.

The Fibonacci 2-numbers $F_2(n)$ and the generalized Leonardo 2-numbers $\mathcal{L}_{2,2}(n)$ are the sequences A000930 and A362255, respectively.

The non-homogeneous recurrence relation of the generalized Leonardo p-numbers can be rewritten as the following homogeneous recurrence relation:

$$\mathcal{L}_{p,k}(n) = \mathcal{L}_{p,k}(n-1) + \mathcal{L}_{p,k}(n-p) - \mathcal{L}_{p,k}(n-2p-1) \quad \text{for } n > 2p.$$

Next, we establish a connection between the generalized Leonardo p-numbers and the Fibonacci p-numbers.

Theorem 2. *For positive integers p, k and a nonnegative integer n, we have*

$$\mathcal{L}_{p,k}(n) = (k+1)F_p(n+1) - k. \tag{1}$$

Proof. It is easy to check that the identity (1) holds for $n = 0, 1, 2, \ldots, p$. Assume that, for some $n \geq p$, the identity (1) holds for all integers m such that $0 \leq m \leq n$. By the definition and the inductive hypothesis, we obtain

$$\begin{aligned}
\mathcal{L}_{p,k}(n+1) &= \mathcal{L}_{p,k}(n) + \mathcal{L}_{p,k}(n-p) + k \\
&= (k+1)F_p(n+1) - k + (k+1)F_p(n-p+1) - k + k \\
&= (k+1)F_p(n+2) - k,
\end{aligned}$$

which shows that the identity (1) holds for $n+1$, thereby proving the theorem. □

Theorem 3. *For positive integers p, k and a nonnegative integer n, we have*

$$\sum_{i=0}^{n} \mathcal{L}_{p,k}(i) = \mathcal{L}_{p,k}(n+p+1) - k(n+1) - 1.$$

Proof. By the definition of $\mathcal{L}_{p,k}(n)$, write

$$\mathcal{L}_{p,k}(i) = \mathcal{L}_{p,k}(i+p+1) - \mathcal{L}_{p,k}(i+p) - k.$$

Summing a telescoping series, we get

$$\sum_{i=0}^{n} \mathcal{L}_{p,k}(i) = \mathcal{L}_{p,k}(n+p+1) - \mathcal{L}_{p,k}(p) - k(n+1).$$

Since $\mathcal{L}_{p,k}(p) = 1$, we obtain the desired result. □

Theorem 4. *Let m and n be nonnegative integers. For positive integers p, k, we have*

$$\sum_{i=0}^{n} \mathcal{L}_{p,k}((p+1)i + m) = \begin{cases} \mathcal{L}_{p,k}\left((p+1)n + m + 1\right) - kn, & \text{if } 0 \leq m < p; \\ \mathcal{L}_{p,k}\left((p+1)(n+1)\right) - k(n+1) - 1, & \text{if } m = p. \end{cases}$$

Proof. Since

$$\mathcal{L}_{p,k}(i(p+1)+m) = \mathcal{L}_{p,k}\left(i(p+1)+m+1\right) - \mathcal{L}_{p,k}\left((i-1)(p+1)+m+1\right) - k,$$

by summing a telescoping series we get

$$\sum_{i=0}^{n} \mathcal{L}_{p,k}(i(p+1)+m) = \mathcal{L}_{p,k}\left((p+1)n+m+1\right) - \mathcal{L}_{p,k}(-p+m) - k(n+1).$$

Since $\mathcal{L}_{p,k}(-i) = -k$ for $1 \leq i \leq p$ and $\mathcal{L}_{p,k}(0) = 1$, the result follows. □

Here are some examples of Theorem 4 where $p = 2$.

(*i*) $\displaystyle\sum_{i=0}^{n} \mathcal{L}_{2,k}(3i) = \mathcal{L}_{2,k}\left(3n+1\right) - kn$

(*ii*) $\displaystyle\sum_{i=0}^{n} \mathcal{L}_{2,k}(3i+1) = \mathcal{L}_{2,k}\left(3n+2\right) - kn$

(*iii*) $\displaystyle\sum_{i=0}^{n} \mathcal{L}_{2,k}(3i+2) = \mathcal{L}_{2,k}\left(3n+3\right) - k(n+1) - 1.$

Moreover, substituting $k = p$ in Theorem 2, Theorem 3 and Theorem 4, we obtain the identities of Theorem 1, Proposition 2 and Theorem 2, in [8], respectively.

3 Main results

In this section, we introduce a companion matrix of the generalized Leonardo p-numbers for $p \geq 2$ (for the case $p = 1$, see [4, page 4]). For a given integer $p \geq 2$, let $H(p)$ consist of all those $(2p+1)^{\text{th}}$ order recurrence sequences $\{\mathcal{H}_p(n)\}_{n\in\mathbb{Z}}$ of real numbers satisfying the relation

$$\mathcal{H}_p(n) = \mathcal{H}_p(n-1) + \mathcal{H}_p(n-p) - \mathcal{H}_p(n-2p-1).$$

Note that the sequences $\mathcal{L}_{p,k}(n)$ are elements of $H(p)$.

Two significant sequences A_p and B_p in $H(p)$ are defined with the following initial values:

(*i*) $A_p(1) = A_p(2) = \cdots = A_p(p) = 1$ and $A_p(0) = A_p(-1) = \cdots = A_p(-p) = 0$,

(*i*) $B_p(1) = B_p(2) = \cdots = B_p(p) = 1$, $B_p(0) = B_p(-1) = \cdots = B_p(-p+2) = 0$, $B_p(-p+1) = 1$ and $B_p(-p) = 0$.

We first derive a relationship between the sequences F_p and A_p.

Theorem 5. *Let n be a positive integer. For positive integers p, we have*

$$F_p(n) = A_p(n) - A_p(n-p). \tag{2}$$

Proof. Observe that

$$F_p(n) = \begin{cases} A_p(n), & \text{if } 0 \leq n \leq p; \\ A_p(n) - 1, & \text{if } p < n \leq 2p. \end{cases}$$

So the identity (2) holds for all integers $n = 0, 1, 2, \ldots, 2p$. Now, assume that for some integer $n \geq 2p$, the identity (2) holds for all integers m such that $0 \leq m \leq n$. By the definition and the inductive hypothesis,

$$\begin{aligned} A_p(n+1) - A_p(n-p+1) &= A_p(n) + A_p(n-p+1) - A_p(n-2p) - A_p(n-p+1) \\ &= A_p(n) - A_p(n-p) + A_p(n-p) - A_p(n-2p) \\ &= F_p(n) + F_p(n-p) \\ &= F_p(n+1), \end{aligned}$$

showing that the identity (2) holds for $n+1$. $\square$

Consequently, we can write $A_p(n)$ as a sum of the Fibonacci p-numbers.

Corollary 6. *Let j be an integer such that $0 \leq j < p$. Then*

$$A_p(mp+j) = \sum_{i=0}^{m} F_p(ip+j). \tag{3}$$

Some special cases of the sequences A_p and B_p and their first few terms are given in the tables.

- For $p = 2$, the first 16 terms of $A_2(n)$ and $B_2(n)$ are

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$A_2(n)$	1	1	2	3	5	7	11	16	24	35	52	76	112	164	241	353
$B_2(n)$	1	1	2	2	4	5	8	11	17	24	36	52	77	112	165	241

- For $p = 3$, the first 17 terms of $A_3(n)$ and $B_3(n)$ are

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$A_3(n)$	1	1	1	2	3	4	6	8	11	16	22	30	42	58	80	111	153
$B_3(n)$	1	1	1	2	2	3	5	6	8	12	16	22	31	42	58	81	111

- For $p = 4$, the first 18 terms of $A_4(n)$ and $B_4(n)$ are

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$A_4(n)$	1	1	1	1	2	3	4	5	7	9	12	16	22	29	38	50	67	89
$B_4(n)$	1	1	1	1	2	2	3	4	6	7	9	12	17	22	29	38	51	67

The sequences A_2, A_4, B_3 and B_4 are the sequences A023435, A233522, A017819 and A017830, respectively.

Some additional properties of the sequences A_p and B_p are given below. The proof is a straightforward induction and shall be omitted.

Proposition 7. *Let n be a nonnegative integer. Then*

$$(i)\ \ A_p(n) = \begin{cases} A_p(n-1) + A_p(n-p-1) + 1, & \textit{if } n \equiv 1 \pmod p; \\ A_p(n-1) + A_p(n-p-1), & \textit{otherwise.} \end{cases}$$

$$(ii)\ \ B_p(n) = \begin{cases} B_p(n-1) + B_p(n-p-1) + 1, & \textit{if } n \equiv 1 \pmod p; \\ B_p(n-1) + B_p(n-p-1) - 1, & \textit{if } n \equiv 2 \pmod p; \\ B_p(n-1) + B_p(n-p-1), & \textit{otherwise.} \end{cases}$$

Next, we derive a relationship between the sequences A_p and B_p.

Proposition 8. *Let $n \geq 0$ be an integer. Then*

$$A_p(n) = \begin{cases} B_p(n+1) - 1, & \textit{if } n \equiv 0 \pmod p; \\ B_p(n+1), & \textit{otherwise.} \end{cases} \tag{4}$$

Proof. It is easy to see that the base cases of the identity (4) hold. Now assume that, for some integer $n \geq 2p$, the identity (4) holds for all integers m such that $0 \leq m \leq n$. By the definition and the inductive hypothesis, we obtain

$$\begin{aligned} A_p(n+1) &= A_p(n) + A_p(n-p+1) - A_p(n-2p) \\ &= \begin{cases} B_p(n+1) - 1 + B_p(n-p+2) - B_p(n-2p+1) + 1, & \text{if } n \equiv 0 \pmod p; \\ B_p(n+1) + B_p(n-p+2) - 1 - B_p(n-2p+1), & \text{if } n \equiv -1 \pmod p; \\ B_p(n+1) + B_p(n-p+2) - B_p(n-2p), & \text{otherwise} \end{cases} \\ &= \begin{cases} B_p(n+2) - 1, & \text{if } n+1 \equiv 0 \pmod p; \\ B_p(n+2), & \text{otherwise,} \end{cases} \end{aligned}$$

thereby proving the theorem. $\square$

Now we proceed to the main result. For $n \geq p \geq 2$, define the $(2p+1) \times (2p+1)$ companion matrix $Q_p = [q_{ij}]$ where

$$q_{ij} = \begin{cases} 1, & \text{if } i = 1, p \text{ and } j = 1, \text{ and } i = j+1; \\ -1, & \text{if } i = 2p+1, j = 1; \\ 0, & \text{otherwise.} \end{cases}$$

For instance, the matrices Q_p for $p = 2, 3, 4$ are as follows:

$$Q_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$Q_4 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Theorem 9. *Let n be a nonnegative integer. Then*

$$Q_p^n = \begin{bmatrix} A_p(n+1) & A_p(n) & A_p(n-1) & \cdots & A_p(n-2p+1) \\ B_p(n-p+2) & B_p(n-p+3) & B_p(n-p+4) & \cdots & B_p(n-3p+2) \\ B_p(n-p+3) & B_p(n-p+4) & B_p(n-p+5) & \cdots & B_p(n-3p+3) \\ \vdots & \vdots & \ddots & & \vdots \\ B_p(n) & B_p(n-1) & B_p(n-2) & \cdots & B_p(n-2p) \\ -A_p(n-p) & -A_p(n-p-1) & A_p(n-p-2) & \cdots & A_p(n-3p) \\ \vdots & \vdots & \ddots & & \vdots \\ -A_p(n-1) & -A_p(n-2) & A_p(n-3) & \cdots & A_p(n-2p-1) \\ -A_p(n) & -A_p(n-1) & A_p(n-2) & \cdots & A_p(n-2p) \end{bmatrix}.$$

Proof. The case $n = 1$ follows from the initial values of the sequences A_p and B_p. Write

$$Q_p^{n+1} = Q_p^n Q_p.$$

By the inductive hypothesis and the definitions of A_p and B_p, a straightforward matrix multiplication shows that the result holds for $n + 1$. □

For positive integers m, define the $(2p + 1) \times (2p + 1)$ matrix $R_{p,m}$ by

$$R_{p,m} = \begin{bmatrix} \mathcal{H}_p(m) & \mathcal{H}_p(m-1) & \mathcal{H}_p(m-2) & \cdots & \mathcal{H}_p(m-2p) \\ \mathcal{H}_p(m-1) & \mathcal{H}_p(m-2) & \mathcal{H}_p(m-3) & \cdots & \mathcal{H}_p(m-2p-1) \\ \mathcal{H}_p(m-2) & \mathcal{H}_p(m-3) & \mathcal{H}_p(m-4) & \cdots & \mathcal{H}_p(m-2p-2) \\ \vdots & \vdots & \ddots & & \vdots \\ \mathcal{H}_p(m-2p) & \mathcal{H}_p(m-2p-1) & \mathcal{H}_p(m-2p-2) & \cdots & \mathcal{H}_p(m-4p) \end{bmatrix}$$

where $\mathcal{H}_p$ is an element of $H(p)$.

Theorem 10. *Let $n \geq 0$ be an integer. For every positive integer m, we have*

$$R_{p,m} Q_p^n = R_{p,m+n}. \tag{5}$$

Proof. The case $n = 0$ is trivial. Now assume that the identity (5) is true for an integer $n \geq 0$. Since

$$R_{p,m}Q_p^{n+1} = \left(R_{p,m}Q_p^n\right)Q_p,$$

by the inductive hypothesis and the definition of $\mathcal{H}_p$, a straightforward matrix multiplication shows that the result holds for $n+1$. $\square$

Equating the $(1,1)$-entry on both sides of the equation (5), we obtain the following corollary.

Corollary 11. *Let $m, n > 0$ be two integers. For a sequence $\mathcal{H}_p \in H(p)$, we have*

$$\mathcal{H}_p(m+n) = \mathcal{H}_p(m)A_p(n+1) + \sum_{i=1}^{p-1}\mathcal{H}_p(m-i)B_p(n-p+i+1) - \sum_{i=0}^{p}\mathcal{H}_p(m-p-i)A_p(n-p+i).$$

Note that, since $\mathcal{L}_{p,k}, A_p, B_p \in H(p)$, the identity of Corollary 11 holds for each of these sequences as well.

Lastly, we derive the analogous version of Honsberger's formula for the generalized Leonardo p-numbers.

Lemma 12. *Let n be a positive integer. Then*

$$(k+1)\sum_{i=0}^{p}F_p(n+i) = \mathcal{L}_{p,k}(n+2p-1) + k.$$

Proof. Using the sum of Fibonacci p-numbers formula (see, e.g., [7])

$$\sum_{i=1}^{n}F_p(i) = F_p(n+p+1) - 1,$$

we get that

$$\begin{aligned}\sum_{i=0}^{p}F_p(n+i) &= \sum_{i=1}^{n+p}F_p(i) - \sum_{i=1}^{n-1}F_p(i)\\ &= F_p(n+2p+1) - F_p(n+p)\\ &= F_p(n+2p).\end{aligned}$$

Using Theorem 2, we obtain

$$(k+1)\sum_{i=0}^{p}F_p(n+i) = (k+1)F_p(n+2p) = \mathcal{L}_{p,k}(n+2p-1) + k,$$

as desired. $\square$

Theorem 13. *Let m and n be two positive integers. Then*

$$\begin{aligned}\mathcal{L}_{p,k}(m)\mathcal{L}_{p,k}(n+1) &+ \sum_{i=1}^{p}\mathcal{L}_{p,k}(m-j)\mathcal{L}_{p,k}(n-p+i)\\ &= (k+1)\mathcal{L}_{p,k}(m+n+1) - k(\mathcal{L}_{p,k}(m+p) + \mathcal{L}_{p,k}(n+p+1)) + pk^2 + k.\end{aligned}$$

Proof. By Theorem 2, write

$$\mathcal{L}_{p,k}(m)\mathcal{L}_{p,k}(n) = (k+1)^2F_p(m+1)F_p(n+1) - k(k+1)(F_p(m+1)+F_p(n+1)) + k^2.$$

Applying Lemma 12 and Honsberger's formula for the Fibonacci p-numbers [9], we have

$$\begin{aligned}&\mathcal{L}_{p,k}(m)\mathcal{L}_{p,k}(n+1) + \sum_{i=1}^{p}\mathcal{L}_{p,k}(m-j)\mathcal{L}_{p,k}(n-p+i)\\ &\quad= (k+1)^2\left(F_p(m+1)F_p(n+2) + \sum_{i=1}^{p}F_p(m-i+1)F_p(n-p+i+1)\right)\\ &\qquad - k(k+1)\left(F_p(m+1)+F_p(n+2) + \sum_{i=1}^{p}F_p(m-i+1)F_p(n-p+i+1)\right)\\ &\qquad + (p+1)k^2\\ &\quad= (k+1)^2F_p(m+n+2) - k(k+1)\left(\sum_{i=0}^{p}F_p(m-i+1) + \sum_{i=1}^{p+1}F_p(n-p+i+1)\right)\\ &\qquad + (p+1)k^2\\ &\quad= (k+1)(\mathcal{L}_{p,k}(m+n+1)+k) - k\left(\mathcal{L}_{p,k}(m+p) + \mathcal{L}_{p,k}(n+p+1) + 2k\right)\\ &\qquad + (p+1)k^2\\ &\quad= (k+1)\mathcal{L}_{p,k}(m+n+1) - k(\mathcal{L}_{p,k}(m+p) + \mathcal{L}_{p,k}(n+p+1)) + pk^2 + k,\end{aligned}$$

as desired. ☐

Substuting $k = p$ in Theorem 13 and using Lemma 12, we obtain Theorem 1 of [8].

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