Empathetic Decision Making in Social Networks☆

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Abstract

Social networks play a central role in the transactions and decision making of individuals by correlating the behaviors and preferences of connected agents. We introduce a notion of *empathy* in social networks, in which individuals derive utility based on both their own intrinsic preferences, and *empathetic preferences* determined by the satisfaction of their neighbors in the network. After theoretically analyzing the properties of our empathetic framework, we study the problem of *group recommendation*, or consensus decision making, within this framework. We show how this problem translates into a weighted form of classical preference aggregation (e.g., social welfare maximization or certain forms of voting), and develop scalable optimization algorithms for this task. Furthermore, we show that our framework can be generalized to encompass other multiagent systems problems, such as constrained resource allocation, and provide scalable iterative algorithms for these generalizations. Our empirical experiments demonstrate the value of accounting for empathetic preferences in group decisions, and the tractability of our algorithms.

Keywords: Social Choice, Empathetic Preferences, Social Networks

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1. Introduction

Social networks facilitate interactions and behaviours between individuals, businesses, and organizations, ranging from discovery of job opportunities [2] and the products we consume [3], to how we vote [4] and how we cooperate [5]. It is widely acknowledged that the behaviors, and to a lesser extent preferences, of individuals connected in a social network are correlated in ways that can be explained, in part, by network structure [6, 7, 8]. Because of this, and the increasing availability of data that allows one to infer such relationships—either directly through online social networks like Facebook, or indirectly through online-mediated communications or transaction data—the study of social choice and group decision making on social networks is of both theoretical and practical importance.

Arguably, most group decision problems, whether social, corporate, or policy-oriented, involve people linked by myriad social ties. These ties may provide strong clues as to the preferences of individuals, which can then be used to facilitate the process of preference aggregation required to implement some *social choice function*, in other words, to make a *group decision*. Indeed, decision making for groups in which relatively few social ties exist (e.g., national elections) is much rarer than decisions where strong ties exist: family and social groups, team formation in companies, local public projects and policy decisions, etc. Even national political elections and policy decisions take place in the context of individuals that are linked by various (local) social ties. Several broad questions guide our investigation of social choice problems in social networks: (i) How are individual preferences shaped by social networks? (ii) How should one mathematically model such processes? (iii) Can such processes be harnessed for more effective group decision making?

We address a specific aspect of these questions by considering the role that *empathy* plays in shaping preferences in the context of social relationships. We introduce a novel *empathetic social choice framework* in which agents derive utility based on both their own *intrinsic preferences* and their *empathetic preferences*, which are determined by the satisfaction of their acquaintances.

We start by focusing on the problem of *group decision making* on social networks. Formally, the goal is to select a single alternative or decision from a set of options for some group connected by a social network, e.g., a local constituency electing a political representative; friends selecting a vacation spot or a movie; selecting a policy for an online system or a nation given individuals' preferences, etc. While individuals have personal *intrinsic utility* over the options, we also incorporate a novel form of *empathetic utility* on social networks: the utility (or satisfaction) of an individual with an alternative a is a function of both her intrinsic utility for a and her *empathetic utility* for the "happiness" of her neighbors with the selected option. Empathetic utility in this sense reflects the fact that a person's happiness may be influenced by the happiness of others with whom they are connected [9].

We consider two varieties of empathetic preference. In our *local empathetic model*, the utility of individual i for alternative a combines her intrinsic preference for a with the *intrinsic* preference of i's neighbors for a, where the weight given to j's preference depends on the strength of the relationship of i with j. For example, when choosing a restaurant, an individual i may be willing to sacrifice a small amount of personal, intrinsic utility to accommodate the preferences of her colleagues, and may be being willing to defer even further if the group consists of close friends. In our *global empathetic model*, i's utility for a depends on her intrinsic preference and the *total utility* of her neighbors for a (not just their intrinsic preference): she wants her neighbors not only to be satisfied with a, but to have high utility, which depends on the utility of *their* neighbors, and so on.

There are a number of real-world scenarios that illustrate this property. For example, in an election an individual i may have a slight preference for candidate a over candidate b, but if b is strongly preferred by her closest neighbors, their neighbors, etc., then she may prefer to see b elected so as to ensure community cohesiveness. In a different scenario, companies linked in complex supply chains may care about the success of their suppliers and customers, and hence consider adopting industry-specific or economic policies in that light.

In the global model, because individual utilities are interdependent—indeed, utility spreads much like PageRank values [10]—it requires the solution of a linear system to determine equilibrium utilities. We describe conditions under which such equilibria or fixed points exist, hence, when empathetic utilities are well-defined. In particular, the conditions capture positive correlations between the utilities of neighbors in social networks (i.e., an individual's utility doesn't decrease when its neighbor's utility increases), and bounded degrees of selflessness (i.e., an individual is not completely indifferent to his/her own intrinsic preferences). These conditions often hold in casual group decisions (e.g., where to eat, vacation, etc.), corporate decisions, or other group-decision making scenarios, where individuals do not derive satisfaction from decreasing utility of others (e.g., jealousy or envy), nor are completely selfless (i.e., do not ignore their own intrinsic or personal preferences).

We devise methods for computing social-welfare maximizing outcomes under both local and global models. We show that social-welfare maximization in both models can be recast as *weighted preference aggregation* over intrinsic preferences alone, where weights are determined by social network structure, but observe that the computation of weights is significantly different in each model. Furthermore, we develop two scalable iterative algorithms for group decision making under the global model.

We run extensive experiments on randomly generated social networks with synthetic and real-world preferences. Our results confirm that neglecting empathy usually yields sub-optimal group decisions which degrade the well-being of group members. For example, in some experimental settings, ignoring empathetic preferences yields decisions that are often suboptimal and give rise to significant social welfare loss. Our experiments also show that, both the distribution of empathetic preferences across the group or population and the structure of the social network play an important role in determining how empathetic preferences influence optimal group decisions. Finally, our experiments demonstrate the computational effectiveness of our algorithms.

After reviewing the related work in Section 2, we outline our empathetic framework for consensus decision making in Section 3. We develop algorithms for this problem in Section 4. We present experiments and empirical analyses in Section 5. Finally, Section 6 concludes this paper and presents possible future research directions.

2. Related Work

The term empathy is used in several different ways in the literature [11]. Sometimes it refers to "seeing the world through the eyes of others" without being affected by this view, and such preferences [12] or "extended sympathy" [13, 14] is used to frame interpersonal comparison of utilities [15, 12]. However, our model is more consistent with an *affective understanding of another*, and having concern for that person's welfare [16], or having "other-regarding" preferences [17]. Empathy has recently drawn attention in neuroeconomics and social neuroscience [18] as a means to explain the extent to which people can place themselves in the position of others and share another's feelings. This further motivates the computational study of empathy and its application to social choice.

The impact that the actions and utilities of others has on an agent is considered in certain economic models (see, e.g., accounts of envy, sympathy/empathy in various contexts [19, 17, 12]). Most closely related to our work is the model of Maccheroni *et al.* [19], who establish the axiomatic foundations of interdependent "other-regarding" preferences in which the outcome experienced by others affects the utility of an agent. In their general formulation, the utility of an agent for an act incorporates both its subjective expected utility for that act and an *expected externalities function* over the agent's perceived social value of its own act and others' acts. While the general form of these externalities can model our notion of empathy, the specific axioms proposed for that model preclude its direct application to our setting. For example, their anonymity axiom prevents the agent from distinguishing which of its peers attains a specific outcome. Furthermore, the work of Maccheroni *et al.* does not deal with group decision making and relevant algorithm development, which is the main focus of our work.

Models of *opinion formation* and *social learning* in social networks are also related (see [8, 20] for a review). Our empathetic model can be viewed mathematically as a special case of a general model of opinion formation due to Friedkin and Johnson [21]. However, their focus is very different than ours. Friedken and Johnson explore the propagation of opinions and beliefs on social networks, while we capture preference interdependence as a form of empathy, and focus on algorithms and mechanisms to implement a given social choice function.

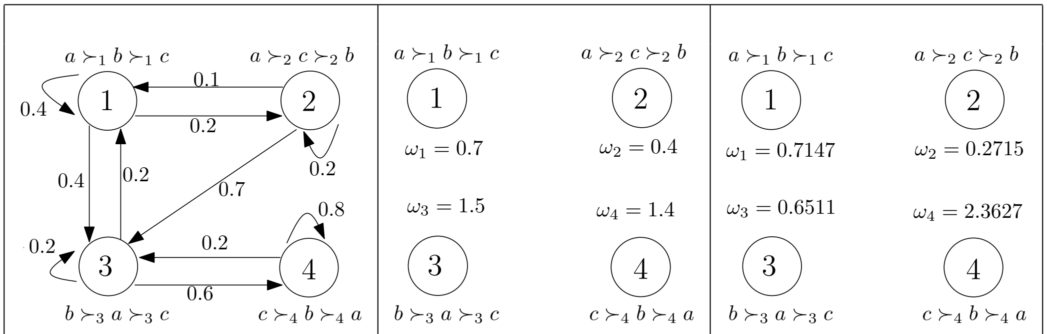
Our empathetic model bears some resemblance to centrality measures in social and information networks which use (self-referential) notions of node *importance*. Some well-known examples include *eigenvector centrality* [22], *hubs and authorities* [23] and *PageRank* [10] (see [7, 8, 24] for a review). Apart from conceptual differences and the fact that we address *decision* problems, a key technical distinction is the use of self-loops in our empathetic model, which allows each node to contribute intrinsic utility to its fixed-point value. We again emphasize that much of the literature on centrality metrics does not address group decision making problems or algorithms.

Empathetic utilities can also be viewed as a form of externality in an agent's utility function, though unlike typical models of *allocative externalities*, an agent's utility depends on the utility of her neighbors for the chosen alternative rather than the behavior of, or the (direct) allocation made to, her neighbors. We formally discuss these distinctions further in Section 3.2. Decision making on social networks in the presence of *allocative externalities* has recently attracted considerable attention. The literature has tackled various computational social choice problems such as stable matching [25], coalition formation [26, 27, 28], voting [29, 30, 31, 32, 33], auction design [34], and resource allocation [35] on social networks in the presence of allocative externalities. In the remainder of this section, we elaborate on this strand of related work. However, again our work differs in its emphasis on the development and analysis of empathetic models for group decision making and corresponding algorithm design.

Bodine-Baron *et al.* [25] study stable matchings (e.g., of students to residences) with *peer effects*: these local social network externalities reflect the fact that students prefer to be assigned to the same residence as their friends. Brânzei and Larson address coalition formation on social networks where an agent's utility for a coalition depends on either her *affinity weights* with others in the coalition [26]; or her closeness centrality measure [27] (closeness centrality measures how close a given node is to other nodes in the network). Hoefer *et al.* [36] have also studied the computational complexity and properties of coalitions on social networks under *considerate equilibria*, in which an agent avoids taking actions that might harm its neighbors. Recently, considerate equilibria have been studied in strategic voting on social networks [37].

Bhalgat *et al.* [35] focus on utilitarian social welfare maximization in unit-demand resource allocation problem in the presence of positive externalities arising from social networks. In their model, each agent's overall utility is the *multiplication* of its *intrinsic valuation* function—mapping each alternative to a utility value—and its *externalities function*. The externalities function maps the number of neighbors with the same assigned alternative to a real value. There has also been growing interest in price setting [38, 39, 40], object swapping [41], and strategic marketing [42, 43] over social networks, which can be viewed as a form of decision making.

Boldi *et al.* [29, 30] study *delegative democracy* on social networks, where an individual can either express her preferences directly, or delegate her vote to a neighbor. The weights of delegated votes are exponentially damped by an attenuation factor that reduces the weight of a vote as it passes from one person to another. In a game-theoretic framework, Alon *et al.* [32] studied the "herding effect" in sequential voting over only two alternatives, with



(a) network & preferences | (b) local empathetic | (c) global empathetic

Figure 1: *A social network with ranked preferences, with weights under the local and global empathetic model.*

agents having private preferences for alternatives and experiencing disutility if the winner is not the alternative for which they vote. Mechanism design for approval voting in social network has been studied by Alon *et al.* [44], where an edge in the underlying social network represents "who approves of whom."

Salehi-Abari and Boutilier [31] study voting on social networks when dealing with missing preferences. They develop probabilistic models [45] and design inference algorithms for making group decisions when some (or all) preferences of group members are unknown but the social network is observed. Their empirical analyses demonstrate that incorporating social ties can significantly improve predictions and group decision making. Tsang *et al.* [33, 46] study strategic voting on social networks with the presence of homophily. By observing their direct neighbors' ballots, voters can strategize their votes. Their analysis suggests that homophily reduces the frequency of strategic voting. See the recent survey by Grandi [47] for further related work of social choice on social networks.

3. Modeling Empathetic Agents in Social Networks

We introduce the basic model of empathetic preferences in social networks, first distinguishing local and global preferences, and then relating our approach to models of allocative externalities.

3.1. Local and Global Empathetic Preferences

We assume a set $\mathcal{N} = \{1, \ldots, n\}$ of agents and a set $\mathcal{A} = \{a_1, \ldots, a_m\}$ of alternatives or outcomes. Agents in $\mathcal{N}$ have preferences over the alternatives $\mathcal{A}$ which are captured by utility functions. Let $u_j : \mathcal{A} \mapsto \mathbb{R}$ be the utility function of agent $j \in \mathcal{N}$. Furthermore, we assume that the agents are *empathetic*, meaning that their utility for that alternative depends on both their intrinsic utility for the alternative, and also that of other agents. We model this by a directed weighted graph $G = (\mathcal{N}, E)$ where an edge (j, k) indicates that j's utility is dependent on its neighbor k's utility, with the strength of the dependence is given by the edge weight. We allow for self-loops; i.e., edges of the form (j, j). We treat missing edges as having weight 0. Thus, we can represent G by a weighted adjacency matrix $\mathbf{W} = [w_{jk}]$.

Figure 1(a) illustrates our empathetic model. There are four agents, 1 through 4, and three alternatives a, b, and c. Each agent has (qualitative) preferences represented as a ranking over the alternatives. For example, agent 1 prefers alternative a to b, and b to c. The ranking preferences can be converted to utilities using any scoring rules (e.g. Borda or Plurality). We note, for example, that the edge between agent 2 and agent 3 has weight 0.7. This means that agent 2's overall utility heavily depends on the utility of agent 3 with respect to the final outcome selected. This example reflects important aspects of various real-world scenarios. For example, imagine a family of four, with different degrees of empathy toward each other, trying to select a restaurant at which to eat. While each individual has a personal preference over the restaurant choice, family members are also influenced by how much the others will like the choice.

We decompose the utility function of each agent j into two components. Given alternative $a \in \mathcal{A}$, agent j derives *intrinsic utility*, $u_j^I(a)$ from the alternative consistent with its underlying intrinsic preferences, and *empathetic utility* $e_{jk}(a)$ from other agents $k \in \mathcal{N}$.[2] In particular, the utility of agent j, given alternative a, is

$$u_j(a) = w_{jj} u_j^I(a) + \sum_{k \neq j} w_{jk} e_{jk}(a). \tag{1}$$

[2]The assumption that preferences can be broken to intrinsic preferences and some form of *exogenous preferences* is explored in the literature on allocative externalities (for example, see [25, 35] and related work therein).

The ratio of w_{jj} to $\sum_{k \neq j} w_{jk}$ captures the relative importance of intrinsic and empathetic utility to j. We note that our framework does not *impose* empathetic preferences on agents: fully self-regarding agents are represented by isolated nodes with self-loops.

An agent's intrinsic preference depends solely on the particular alternative under consideration, and is independent of the preferences of its neighbors. One can easily distinguish between intrinsic preferences and empathetic preferences by observing how an agent's preferences change when they are in a social setting compared to when they are in isolation. For example, an individual, knowing that they will watch a movie alone, may reveal an intrinsic preference for horror films, while in a particular group setting they may prefer action movies. Intrinsic preferences can also be elicited through well-formed elicitation queries or with observations of how an agent interacts with the underlying set of alternatives in different contexts, particularly when acting independently.

Let $\mathbf{u} = (u_1(\cdot), \ldots, u_n(\cdot))$ be a vector specifying the utility function of each agent. Given $\mathbf{u}$, the (utilitarian) *social welfare* of alternative a is:

$$sw(\mathbf{u}, a) = \sum_j u_j(a). \tag{2}$$

We consider two different empathetic models. In the *local empathetic model*, we let $e_{jk}(a) = u_j^I(a)$, implying that given alternative a, the utility of agent j is a weighted sum of its own and its neighbors' intrinsic utilities. That is

$$u_j(a) = \sum_{k \in \mathcal{N}} w_{jk} u_j^I(a). \tag{3}$$

Equivalently, if we define $\mathbf{u}(a)$ to be the n-vector of agent utilities for outcome a and $\mathbf{u}^I(a)$ the n-vector of intrinsic utilities for the outcome, then

$$\mathbf{u}(a) = \mathbf{W}\mathbf{u}^I(a) \tag{4}$$

where $\mathbf{W}$ is the weight matrix defined above. Defining

$$\boldsymbol{\omega}_l^\top = \mathbf{e}^\top \mathbf{W}, \tag{5}$$

where $\mathbf{e}$ is a column vector of ones of suitable dimension, the *local* social welfare of alternative a is

$$sw_l(\mathbf{u}, a) = \boldsymbol{\omega}_l^\top \mathbf{u}^I(a). \tag{6}$$

In other words, local social welfare maximization can be expressed as the weighted maximization of intrinsic preferences, where the weight of agent j's intrinsic utility is the sum of its incoming edge weights. Figure 1(b) shows the weights for each agent computed under the local empathetic model for the network in Figure 1(a). The local empathetic model captures scenarios where an agent is concerned about the direct preferences of its neighbors. For example, when choosing a movie for a family movie-night, the utility a parent derives from a movie choice may depend heavily on how entertaining the children find the movie.

In the *global empathetic model*, we define $e_{jk}(a) = u_k(a)$ so that agent k's overall utility for an alternative a—which may depend on k's neighbors—influences j's utility. This gives rise to

$$u_j(a) = w_{jj}u_j^I(a) + \sum_{k \neq j} w_{jk}u_k(a). \tag{7}$$

In the notation of Equation 4, and defining $\mathbf{D}$ to be an $n \times n$ diagonal matrix with $d_{jj} = w_{jj}$, we have

$$\mathbf{u}(a) = (\mathbf{W} - \mathbf{D})\mathbf{u}(a) + \mathbf{D}\mathbf{u}^I(a). \tag{8}$$

This model captures scenarios where an agent's utility depends on the total *utility* of her neighbors, not just their intrinsic preferences, which in turn depends on the utilities of *their* neighbors, an so on. For example, companies linked in a complex supply chain may care about the success of their suppliers and customers, and so may consider adopting policies that lead to outcomes which are beneficial to all.

One challenge with our formulation of global empathetic utility is that Equation 8 may not have a unique solution. This is illustrated with the simple example of two agents, 1 and 2, with $w_{11} = w_{22} = 0$, $w_{12} = w_{21} = 1$ and any intrinsic utility functions. This results in a continuum of solutions where $u_1(a) = u_2(a)$. However, if the underlying graph G has some additional properties then uniqueness of global empathetic utilities can be guaranteed.

Property 1. ***(Non-negativity)*** *Given graph $G = (\mathcal{N}, E)$, its weight matrix $\mathbf{W}$ is* non-negative *if $w_{jk} \geq 0$ for all $j, k \in \mathcal{N}$.*

The implication of non-negativity is that an individual agent's utility cannot degrade as the utilities of other agents in the system improve. While this property means that our model can not capture all group dynamics, particularly those where individuals are spiteful or envious of others, the space of decision-making scenarios where this property does hold is still very rich. Some problems from the resource-allocation literature that focus on social-welfare maximization satisfy this property, as well as group decision-making scenarios among agents who are not negatively disposed toward each other, as one would expect among functional groups of friends, colleagues, or neighbors. Non-negativity is satisfied in the family decision making example in Figure 1.

Property 2. ***(Positive Self-loop)*** *Given graph $G = (\mathcal{N}, E)$, its weight matrix $\mathbf{W}$ satisfies* positive self-loop *if $w_{jj} > 0$ for all $j \in \mathcal{N}$.*

This property ensures that agents are not entirely selfless—they do care about their own intrinsic utilities to some extent. This condition is a natural constraint and is often satisfied in decision-making scenarios where agents cannot opt out. For example, in casual group decisions like restaurant choice, an individual derives some intrinsic value from the restaurant itself; or in supply-chain settings, a firm experiences direct ramifications of particular decisions, no matter how it affects other firms. Positive self-loop is satisfied in Figure 1.

Property 3. ***(Normalization)*** *Given graph $G = (\mathcal{N}, E)$, its weight matrix $\mathbf{W}$ is* normalized *if $\sum_{k \in \mathcal{N}} w_{jk} = 1$ for all $j \in \mathcal{N}$.*

As long as weights are bounded from above and there are positive self-loops, it is always possible to normalize any $\mathbf{W}$. The example in Figure 1 satisfies this property.

Proposition 3.1. *Let $G = (\mathcal{N}, E)$ with weight matrix $\mathbf{W}$. If $\mathbf{W}$ is non-negative, normalized and satisfies positive self-loop, then the linear system*

$$\mathbf{u}(a) = (\mathbf{W} - \mathbf{D})\mathbf{u}(a) + \mathbf{D}\mathbf{u}^I(a), \quad \forall a \in \mathcal{A},$$

has a unique solution for $\mathbf{u}$, where $\mathbf{D}$ is an $n \times n$ diagonal matrix with $d_{jj} = w_{jj}$ and $\mathbf{u}^I(a)$ is the n-vector of intrinsic utilities for the outcome a. In particular,

$$\mathbf{u}(a) = (\mathbf{I} - \mathbf{W} + \mathbf{D})^{-1}\mathbf{D}\mathbf{u}^I(a), \quad \forall a \in \mathcal{A}.$$

Proofs of all main results can be found in Appendix Appendix B.

As in the case of local social welfare, computing maximum *global social welfare* can also be seen as a weighted maximization of intrinsic preferences.

	Social Welfare		
Alternative	Intrinsic Utility	Local Empathetic	Global Empathetic
a	**5**	3.3	2.6235
b	4	**5.1**	4.3796
c	3	3.2	**4.9969**

Table 1: *Social welfare of alternatives under different models of empathy. The numbers in bold correspond to the optimal social welfare under the different models.*

Corollary 3.1. *Given $G = (\mathcal{N}, E)$ with weight matrix $\mathbf{W}$, if $\mathbf{W}$ is non-negative, normalized and satisfies positive self-loop, then the global social welfare of alternative a is given by*

$$sw_g(\mathbf{u}, a) = \boldsymbol{\omega}_g^\top \mathbf{u}^I(a) \tag{9}$$

where

$$\boldsymbol{\omega}_g^\top = \mathbf{e}^\top (\mathbf{I} - \mathbf{W} + \mathbf{D})^{-1} \mathbf{D}. \tag{10}$$

Figure 1(c) shows the weights for each agent computed under the global empathetic model, given the network in Figure 1(a).

For decision making, we are typically interested in alternatives or outcomes that maximize utilitarian social welfare and this is our focus. For a given $\mathbf{u}$, define the optimal alternative to be

$$a^* = \arg\max_{a \in \mathcal{A}} sw(\mathbf{u}, a). \tag{11}$$

While we emphasize utilitarian social welfare in this work, our empathetic models can be applied to other social welfare functions or objectives readily.

The importance of accounting for empathy can be illustrated by a simple example where the social-welfare maximizing alternative changes depending on whether empathetic preferences are modeled appropriately. Consider the agents in Figure 1(a), and furthermore assume that their intrinsic utility functions are linear; specifically, an agent's most preferred alternative has a utility of two, its second-most preferred alternative has utility one and its least preferred has utility zero. Thus, for example, $u_1^I(a) = 2$, $u_1^I(b) = 1$ and $u_1^I(c) = 0$. Table 1 shows the utilitarian social welfare of each alternative if only intrinsic utilities are used (first column), if a local empathetic model is used (second column), and if a global empathetic model is used (third column). Social welfare for the local and global models are calculated using

Equations 6 and 9, respectively, with weights shown in Figures 1(b) and 1(c). Not only does the social welfare for the different alternatives vary with the empathetic model used, *but the social-welfare maximizing choice differs in each model*. This illustrates that if empathetic preferences are not modeled appropriately and fail to reflect the true preferences of individuals, the optimal (i.e., social-welfare maximizing) decision under the model used will generally differ from the social-welfare maximizing option relative to these true preferences.

We place no particular restrictions on the alternative or outcome space, $\mathcal{A}$, and thus we can use our empathetic model to capture a wide range of applications. For example, in a group-recommendation problem or a voting problem, our alternative space $\mathcal{A}$ consists of the different alternatives or candidates which, once chosen, would be imposed on the group of agents. On the other hand, in allocation problems, $\mathcal{A}$ is generally defined by the space of (feasible) allocations $\mathbf{a} = (a^1, \dots, a^n)$ where a^i is the set of items assigned to agent i. In such a setting, agent i's intrinsic utility function would capture its valuation for a^i, with no further adjustments required to the empathetic model itself. For example, given $\mathbf{a} = (a^1, \dots, a^n)$ where each a^i is the allocation to agent i, we define the global empathetic utility of an agent j as

$$u_j(\mathbf{a}) = w_{jj} u_j^I(a^j) + \sum_{k \neq j} w_{jk} u_k(\mathbf{a}).$$

Thus, as above, social welfare for a particular allocation a takes the form $sw_l(\mathbf{u}, \mathbf{a}) = \boldsymbol{\omega}_l^\top \mathbf{u}^I(\mathbf{a})$ and $sw_g(\mathbf{u}, \mathbf{a}) = \boldsymbol{\omega}_g^\top \mathbf{u}^I(\mathbf{a})$ for local and global empathy, respectively, where $\boldsymbol{\omega}_l$ and $\boldsymbol{\omega}_g$ are defined in Equation 5 and Equation 10, respectively, and $\mathbf{u}^I(\mathbf{a}) = \big(u_1^I(a^1), \cdots, u_n^I(a^n)\big)$ reflects intrinsic allocation utility.

3.2. Empathetic Preferences and Allocative Externalities

Empathetic utilities can be viewed as a form of externality on an agent's utility function. But unlike typical models of *allocative externalities*, an agent's utility depends on the utility of her neighbors for the chosen alternative rather than the behavior of, or the (direct) allocation made to, her neighbors. More specifically, individual i's extrinsic utility in our empathetic model derives from i's neighbors' utilities, whereas in the allocative externalities model, extrinsic utilities are often determined by similarity/dissimilarity (or distance) between i's assignment and each of its neighbors' assignments.

To crystallize this distinction, we show that the empathetic model cannot be subsumed by a general allocative externalities model.

We adopt the model of *metric labeling* as a general model for allocative externalities [48].[3] This model is also used for studying strategic behaviors in the presence of allocative externalities [49], and generalizes other allocative-externality models such as that of Bodine-Baron *et al.* [25].

Definition (**Generalized Allocative Model.**) The utility of individual i for allocation $\mathbf{x} = (x^1, \ldots, x^n)$ under the metric labeling model can be written as

$$u_i(\mathbf{x}) = w_{ii} u_i^I(x^i) + \sum_j w_{ij} u'(x^i, x^j), \tag{12}$$

where x^i represents the assignment (or label) of individual i, $u_i^I(.)$ is i's intrinsic utility function, w_{ij} is the weight of the edge between i and j, and $u'(x^i, x^j)$ is its extrinsic utility. Letting $\beta_0 \in \mathbb{R}$ and $\beta_1 \in \mathbb{R}^+$ be constants, the extrinsic utility $u'(x^i, x^j) = \beta_0 - \beta_1 d(x^i, x^j)$ is the linear function of the distance between x^i and x^j, denoted by $d(x^i, x^j)$.

The closer x^i and x^j, the greater the corresponding extrinsic utility i realizes for agent j's allocation. Note that $d(.,.)$ is a valid distance metric satisfying *non-negativity* $d(y, z) \geq 0$, *identity of indiscernibles* $d(y, z) = 0 \Leftrightarrow y = z$, and *symmetry* $d(y, z) = d(z, y)$.

As an example of generality of this model, the model of Bodine-Baron *et al.* [25] can be derived by using the discrete metric in the model and letting $\beta_0 = 0$ and $\beta_1 = 1$.

Proposition 3.2. *The empathetic model is not subsumed by the metric labeling model.*

4. Computing Social-Welfare Maximizing Outcomes

Given our general framework for representing empathetic preferences, a key question is how to use it effectively to make group decisions, specifically,

[3] The metric labeling problem is typically formulated based on a node's costs and pairwise disagreement penalties/costs. However, we can represent this model equivalently using the concept of utility to be consistent with the terminology used in this paper.

to compute social-welfare maximizing alternatives that account for empathy.[4] Recall that, in both the local and global empathetic models, finding the socially optimal outcome is computationally equivalent to a weighted social welfare maximization problem. The computation of weights in the local model is computationally inexpensive, while weight computation and social-welfare maximization are more computationally challenging in the global model. To address this, we propose two new algorithms for finding global-empathetic optimal outcomes.

Recall that the social welfare of alternative a is defined as $sw(\mathbf{u}, a) = \boldsymbol{\omega}^\top \mathbf{u}^{\boldsymbol{I}}(a)$ where the definition of $\boldsymbol{\omega}$ depended on whether we were interested in local or global empathy. Thus, given intrinsic utility functions $\mathbf{u}^{\boldsymbol{I}}$ and $\boldsymbol{\omega}$, the optimal alternative is

$$a^* = \arg\max_{a \in \mathcal{A}} \boldsymbol{\omega}^\top \mathbf{u}^{\boldsymbol{I}}(a). \tag{13}$$

If $\boldsymbol{\omega}$ is given, the alternative a^* can be found in $O(nm)$ time where n is the number of agents and m is the number of alternatives. We note that, while $O(nm)$ provides an accurate reflection of complexity in some domains (e.g., voting problems where the set of alternatives is explicitly enumerated), when the alternative space $\mathcal{A}$ is combinatorial (e.g., allocation and matching problems), this direct optimization approach may not be viable since the number of alternatives m is itself exponential (or worse) is certain domain features.

The computation of $\boldsymbol{\omega}$ itself is potentially computationally expensive, especially in the global empathetic model. We focus on two approaches in the development of our algorithms for empathetic optimization. In the first, we decouple the computation of $\boldsymbol{\omega}$ from the underlying optimization problem and develop efficient algorithms for weight computation under both the local and global models. These weight-computation algorithms can be viewed as a pre-processing step for any social-welfare maximization algorithms (in Equation 13). In the second approach, we develop an algorithm for computing a^* without explicitly computing $\boldsymbol{\omega}$. This algorithm is well-suited for problems with tractably-enumerable alternative sets.

[4] Although we focus on maximizing social welfare, our general empathetic framework can be adapted easily to maximizing egalitarian welfare or the Nash product. This would require some additional algorithmic development of course. These social welfare functions might be better-suited to group decision making problems in which fairness is a primary concern.

4.1. Local Empathetic Preferences

Recall that under the local model, $\boldsymbol{\omega} = \boldsymbol{\omega}_l = \mathbf{e}^\top \mathbf{W}$ (see Equation 5). Thus, computing $\boldsymbol{\omega}^\top$ requires only a single vector-matrix multiplication, taking $O(n^2)$ time. However, social networks are generally sparse, with the number of incoming edges to any node j typically bounded by some small constant. In such sparse networks, $\boldsymbol{\omega}$ can be computed in time $O(n)$ since ω_j is simply the sum of j's incoming edge-weights; hence, a^* can be determined in time $O(nm)$. Thus the complexity of finding social-welfare maximizing alternatives under the local empathetic model is often no different than any weighted social welfare computation.

4.2. Global Empathetic Preferences

In the global empathetic model, the social welfare problem uses weights

$$\boldsymbol{\omega}^\top = \boldsymbol{\omega}_g^\top = \mathbf{e}^\top \mathbf{A}^{-1}\mathbf{D},$$

where $\mathbf{A} = \mathbf{I} - \mathbf{W} + \mathbf{D}$ is as described in Corollary 3.1. The difficulty here lies largely in matrix inversion. $\mathbf{A}^{-1}$ can be computed via Gauss-Jordan elimination which has complexity $O(n^3)$, implying that a straightforward computation of the optimal alternative requires $O(n^3 + nm)$ time. In general, matrix inversion is no harder than matrix multiplication [50, Theorem 28.2], but its complexity cannot be less than $O(n^2)$ since all n^2 entries must be computed. Therefore, straightforward computation of a^* in the global model cannot have complexity less than $O(n^2 + nm)$.

We expect n to be extremely large in many social choice problems on social networks, e.g., in the tens of thousands (number of people in a small town), the millions (large cities), or hundreds of millions (large country, number of Facebook or Twitter users). Examples of such social choice problems on social networks include local constituencies electing a political representative; a company deciding on assigning an advertisement to a group of users based on the relevance of ads and user preferences; selecting a privacy policy for an online system; or selecting economic or health-care policies for a nation.

For large n, algorithms that scale linearly (or better) in n are needed. Many iterative methods have been proposed for matrix inversion and solving linear systems (e.g., Jacobi, Gauss-Siedel, etc.) which have $O(n)$ per-iteration complexity in sparse systems and tend to converge very quickly in practice (see [51] for an overview). In the rest of this section we investigate iterative methods which can be used for both finding social-welfare maximizing alternatives and computing the social-weight matrix $\boldsymbol{\omega}_g$.

4.2.1. Finding Social Welfare Maximizing Outcomes

We first introduce an algorithm for finding the social-welfare maximizing alternative without directly computing the social weight matrix ω_g. In particular, we use an iterative method for computing $\mathbf{u}(a)$. Let $\mathbf{u}^{(t)}(a)$ be the vector of estimated utilities for alternative a after t iterations, with $\mathbf{u}^{(0)}(a)$ being arbitrary.

Theorem 4.1. *Given graph $G = (\mathcal{N}, E)$ with weight matrix $\mathbf{W}$, alternative set $\mathcal{A}$ and intrinsic utilities $\mathbf{u}^I$, consider the following iteration:*

$$\mathbf{u}^{(t+1)}(a) = (\mathbf{W} - \mathbf{D})\mathbf{u}^{(t)}(a) + \mathbf{D}\mathbf{u}^I(a). \tag{14}$$

Assuming non-negativity, normalization, and positive self-loop, this method converges to $\mathbf{u}(a)$, the solution to Equation 8.

For each $j \in \mathcal{N}$, the method computes:

$$u_j^{(t+1)}(a) = w_{jj}u_j^I(a) + \sum_{k \neq j} w_{jk}u_k^{(t)}(a), \tag{15}$$

where $u_j^{(t)}(a)$ is agent j's estimated utility for a after t iterations. This scheme has a natural interpretation in terms of agent behavior—we can suppose that each agent repeatedly observes her friends' revealed utilities, and updates her own utility for various options in response. This process converges, even if the updates are asynchronous. Under this iterative process, the local empathetic model provides a first-order approximation to the global model if we simply let $u_k^{(0)}(a) = u_k^I(a)$. In other words, with this initialization, after the first iteration, we have computed the utilities for the local model. Critically, the error in the estimated utilities at the t^{th} iteration can also be bounded:

Theorem 4.2. *In the iterative scheme described in Theorem 4.1,*

$$\left\|\mathbf{u}(a) - \mathbf{u}^{(t)}(a)\right\|_\infty \leq (1 - \tilde{w})^t \left\|\mathbf{u}(a) - \mathbf{u}^{(0)}(a)\right\|_\infty,$$

where $\tilde{w} = \min_{1 \leq i \leq n} w_{ii}$.

Here we use the supremum norm $\|\mathbf{A}\|_\infty = \max\limits_{1\le i\le m} \sum_{j=1}^n |a_{ij}|$. This theorem shows that the approximation error at the t^{th} iteration is bounded above by the product of the initial approximation error $\left\|\mathbf{u}(a) - \mathbf{u}^{(0)}(a)\right\|_\infty$ and a damping factor $(1-\tilde{w})^t$ which depends on the minimum self-loop weight in the system. In particular, systems in which individuals have high self-loop weights (and, thus, exhibit less empathy) will converge faster than systems with greater empathy.

The bound in Theorem 4.2 allows us to bound the error in *estimated social welfare* if the utilities of all options are estimated this way. Let $sw^{(t)}(a) = \sum_j u_j^{(t)}(a)$.

Theorem 4.3. *Assume $u_j^I(a), u_j^{(0)}(a) \in [c,d]$, for all j where $c \le d$ are constants. If the properties of normalization, non-negativity, and positive self-loop hold, then $\left|sw(a) - sw^{(t)}(a)\right| \le n(d-c)\,(1-\tilde{w})^t$, for all t, where $\tilde{w} = \min_{1\le i\le n} w_{ii}$.*

An immediate consequence of this result allows us to compare the social welfare of different alternatives.

Proposition 4.4. *Under normalization, non-negativity, and positive self-loop, if*

$$sw^{(t)}(b) - sw^{(t)}(a) \ge 2n(d-c)\,(1-\tilde{w})^t \tag{16}$$

then $sw(b) > sw(a)$.

We exploit Proposition 4.4 in *Iterated Candidate Elimination (ICE)*, a simple algorithm for computing a^*. The intuition underlying ICE is straightforward: we iteratively update the estimated utilities of the subset $C \subset \mathcal{A}$ of options that are non-dominated, and gradually prune away any options that are dominated by another until only one, a^*, remains.

The pseudo-code for the algorithm is provided in Algorithm 1 ICE first initializes the set of non-dominated alternatives to be $C = \mathcal{A}$, and sets $u_j^{(0)}(a) = c$ for all $j \in \mathcal{N}, a \in \mathcal{A}$, where c is a lower bound on intrinsic utility. An iteration of ICE consists of:

1. updating estimated utilities using Equation 15 for all $j \in \mathcal{N}$ and $a \in C$;
2. computing estimated social welfare of each $a \in C$;
3. determining the maximum estimated social welfare $\hat{sw}^{(t)}$;

Algorithm 1: Iterated Candidate Elimination (ICE)

input : Graph G, intrinsic utilities $u_i^I(a)$, $\forall i \in \mathcal{N}$, $\forall a \in \mathcal{A}$.
output: Alternative a^*.

```
Initialize u_i^(0)(a) ← c, ∀i ∈ N and ∀a ∈ A;
// C is the possible winner candidate set
C ← A;
w̃ = min_{1≤i≤n} w_ii;
t ← 0;
while size(C) > 1 do
    t ← t + 1;
    foreach a ∈ C do
        sw^(t)(a) ← 0;
        foreach j ∈ N do
            u_j^(t)(a) ← w_jj u_j^I(a) + Σ_{k:(j,k)∈E, j≠k} w_jk u_k^(t-1)(a);
            sw^(t)(a) ← sw^(t)(a) + u_j^(t)(a);
    sŵ^(t) ← max_{a∈C} sw^(t)(a);
    foreach a ∈ C do
        if sŵ^(t) − sw^(t)(a) ≥ 2n(d − c)(1 − w̃)^t then
            C ← C − {a}
return a* ∈ C
```

4. testing each $a \in C$ for domination, i.e.,

$$\hat{sw}^{(t)} - sw^{(t)}(a) \geq 2n(d-c)\left(1-\tilde{w}\right)^t;$$

5. eliminating all dominated options from C.

The algorithm terminates when one option a^* remains in C.[5] ICE runs in $O(tm|E|)$ time, where t is the number of iterations required; and if the number of outgoing edges at any node is bounded by some constant, running time is $O(tmn)$. As we demonstrate in our experiments, ICE converges quickly in

[5]The correctness (and termination) of ICE relies on the assumption that there is a unique social-welfare maximizing alternative. In practice, this assumption is satisfied for large groups with high probability. If this does not hold, we can limit the maximum number of iterations.

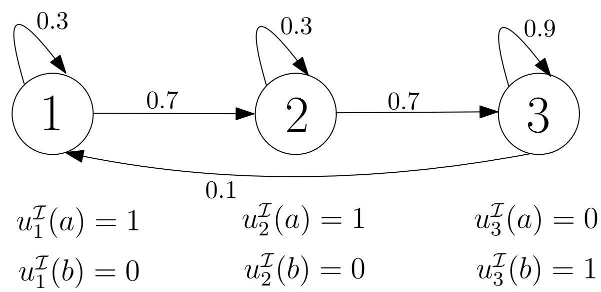


Figure 2: A social network with agents' intrinsic utilities.

$t =$	0	1	2	3	4	5
$u_1^{(t)}(a)$,$u_1^{(t)}(b)$	0,0	0.3,0	0.51,0	0.51,0.441	0.525,0.441	0.535,0.441
$u_2^{(t)}(a)$,$u_2^{(t)}(b)$	0,0	0.3,0	0.3,0.63	0.321,0.63	0.336,0.63	0.336,0.661
$u_3^{(t)}(a)$,$u_3^{(t)}(b)$	0,0	0,0.9	0.03,0.9	0.051,0.9	0.051,0.944	0.052,0.944
$sw^{(t)}(a)$,$sw^{(t)}(b)$	0,0	0.6,**0.9**	0.84,**1.53**	0.882,**1.971**	0.911,**2.015**	0.923,**2.046**
$\hat{sw}^{(t)} - sw^{(t)}(a)$	—	0.3	0.69	1.089	1.103	1.123
$\hat{sw}^{(t)} - sw^{(t)}(b)$	—	0	0	0	0	0
$6(0.7)^t$	6	4.2	2.94	2.058	1.4406	1.00842

Table 2: The computation steps for the ICE algorithm over the example presented in Figure 2. The last row of the table corresponds to the right-hand side of Equation 16.

practice. Referring to Theorem 4.2, we see that the number of iterations t depends on the structure of social network (specially, the minimum self-loop weight) and the distribution of intrinsic preferences.

We illustrate ICE using the simple network in Figure 2. First, ICE initializes $\tilde{w} = 0.3$ and $C = \{a, b\}$. Then, the algorithm iteratively computes the estimated utilities and social welfare, as illustrated in Table 2, until $|C| = 1$. After 5 iterations, the elimination condition for candidate a is met: $\hat{sw}^{(5)} - sw^{(5)}(a) > 2n(d-c)\left(1-\tilde{w}\right)^5$ where $2n(d-c)\left(1-\tilde{w}\right)^5 = 6(0.7)^5$. Thus candidate a is eliminated from C and the algorithm terminates by outputting b as the winner. We note that the algorithm did not compute the social welfare of the alternatives directly, but still successfully found the optimal alternative.

4.2.2. Computing Social Weights

The ICE algorithm finds the social-welfare maximizing alternative without computing the social weight vector directly. However, there are scenarios where the social weights themselves are of interest. In addition, as discussed below, it may be more effective to compute social weights *a priori* and use these to directly find optimal alternatives. In this section, we propose methods for computing weights, and develop *Weight-Based Iterative Candidate Elimination (WICE)*, an algorithm that uses these weights to compute the social-welfare maximizing alternative.

We first note that the problem of weight computation involves solving a linear system of equations:

Theorem 4.5. *The vector* $\boldsymbol{\omega}_g$ *is the unique solution to the linear system of* $\mathbf{A}\boldsymbol{\omega}_g = \mathbf{e}$ *where* $\mathbf{A} = (\mathbf{I} - \mathbf{W}^\top + \mathbf{D})\mathbf{D}^{-1}$.

We now briefly describe a standard Jacobi iterative method for estimating weights $\boldsymbol{\omega}_g$ in the global model. Let $\boldsymbol{\omega}_g^{(t)}$ be the estimated weights after t iterations ($\boldsymbol{\omega}_g^{(0)}$ is arbitrary).

Theorem 4.6. *Given graph* $G = (\mathcal{N}, E)$ *with weight matrix* $\mathbf{W}$, *alternative set* $\mathcal{A}$ *and intrinsic utilities* $\mathbf{u}^I$, *consider the following update:*

$$\boldsymbol{\omega}_g^{(t+1)} = \mathbf{D}(\mathbf{W}^\top - \mathbf{D})\mathbf{D}^{-1}\boldsymbol{\omega}_g^{(t)} + \mathbf{De}$$

Assuming non-negativity, normalization, and positive self-loop, this method converges to $\boldsymbol{\omega}_g$, *the solution to linear system stated in Theorem 4.5.*

For each $j \in \mathcal{N}$, this iterative method computes

$$\omega_j^{(t+1)} = w_{jj} + \sum_{k \neq j} \frac{w_{jj}}{w_{kk}} w_{kj} \omega_k^{(t)}(a), \tag{17}$$

where $\omega_j^{(t)}$ is the agent j's estimated societal weight after t iterations. One can readily bound the error of the estimated weights after t iterations:

Theorem 4.7. *Assume* $\boldsymbol{\omega}_g^{(0)} = (w_{11}, w_{22}, \ldots, w_{nn})^\top$. *In the iterative scheme of Theorem 4.6,*

$$\left\|\boldsymbol{\omega}_g - \boldsymbol{\omega}_g^{(t)}\right\|_1 \leq n\frac{\hat{w}}{\tilde{w}} (1 - \tilde{w})^t (1 - \bar{w}),$$

where $\tilde{w} = \min_{1 \leq j \leq n} w_{jj}$, $\hat{w} = \max_{1 \leq j \leq n} w_{jj}$, *and* $\bar{w} = \frac{1}{n} \sum_j w_{jj}$.

Here, we use the 1-norm $\|\mathbf{A}\|_1 = \max\limits_{1\le j\le n} \sum_{i=1}^{m} |a_{ij}|$. This iterative method converges faster for societies in which individuals have self-loops with relatively larger weight (i.e., less empathy) compared to societies with greater empathy.

This error bound on $\boldsymbol{\omega}_g$ allows one to bound the error in the *estimated* social welfare of any $a \in \mathcal{A}$:

$$sw^{(t)}(a) = \sum_j \omega_j^{(t)} u_j^I(a).$$

Theorem 4.8. *Assume $\boldsymbol{\omega}^{(0)} = (w_{11}, w_{22}, \ldots, w_{nn})^\top$. Under normalization, non-negativity, and positive self-loop, for any t:*

$$|sw(a) - sw^{(t)}(a)| \le n\frac{\hat{w}}{\tilde{w}}\left(1-\tilde{w}\right)^t\left(1-\bar{w}\right)\left\|\mathbf{u}^I(a)\right\|_2,$$

where $\tilde{w} = \min_{1\le j\le n} w_{jj}$, $\hat{w} = \max_{1\le j\le n} w_{jj}$, and $\bar{w} = \frac{1}{n}\sum_j w_{jj}$.

This bounds the error in estimated social welfare at iteration t. As t grows, the error shrinks since $1-\tilde{w} < 1$. Due to having n and $\left\|\mathbf{u}^I(a)\right\|_2$ on the right side, for larger n, we require a greater number of iterations to obtain reasonable approximations. As a result of Theorem 4.8, we know that:

Proposition 4.9. *Under normalization, non-negativity, and positive self-loop, for any t, if*

$$sw^{(t)}(a) - sw^{(t)}(b) \ge n\frac{\hat{w}}{\tilde{w}}\left(1-\tilde{w}\right)^t\left(1-\bar{w}\right)\left(\left\|\mathbf{u}^I(a)\right\|_2 + \left\|\mathbf{u}^I(b)\right\|_2\right), \tag{18}$$

then $sw(a) > sw(b)$, for $a, b \in \mathcal{A}$.

Using this proposition, by comparing the estimated social welfare of two alternatives, one can assess the relative magnitude of their actual social welfare. We exploit this below.

To demonstrate the practical impact of these theoretical results, our experiments in the next section show how they can be used to solve a simplified allocation problem. Moreover, we can use this weight approximation in another iterative algorithm for consensus decision making. We call this new algorithm *weight-based iterated candidate elimination (WICE)*. The intuition behind WICE is to iteratively update the estimated weights $\boldsymbol{\omega}^{(t)}$ and accordingly calculate the estimated social welfare of the subset $C \subset \mathcal{A}$ of

Algorithm 2: Weight-based Iterated Candidate Elimination (WICE)

input : Graph G, intrinsic utilities $u_i^I(a)$, $\forall i \in \mathcal{N}$, $\forall a \in \mathcal{A}$.
output: Alternative a^*.
Initialize $\omega_j^{(0)} \leftarrow w_{jj}$, $\forall j \in \mathcal{N}$;
`//` C `is the possible winner candidate set`
$C \leftarrow \mathcal{A}$;
$\tilde{w} = \min_{1\leq j\leq n} w_{jj}$;
$\hat{w} = \max_{1\leq j\leq n} w_{jj}$;
$\bar{w} = \frac{1}{n}\sum_j w_{jj}$;
$t \leftarrow 0$;
while ***size***$(C) > 1$ **do**
  $t \leftarrow t+1$;
  foreach $j \in \mathcal{N}$ **do**
    $\omega_j^{(t)} \leftarrow w_{jj} + \sum_{k\neq j:(k,j)\in E} \frac{w_{jj}}{w_{kk}} w_{kj}\omega_k^{(t-1)}$;
  foreach $a \in C$ **do**
    $sw^{(t)}(a) = \left(\boldsymbol{\omega}^{(t)}\right)^\top \mathbf{u}^I(a)$
  $\hat{a} = \arg\max_{a\in C} sw^{(t)}(a)$;
  $\hat{sw}^{(t)} \leftarrow sw^{(t)}(\hat{a})$;
  foreach $a \in C$ **do**
    if $\hat{sw}^{(t)} - sw^{(t)}(a) \geq n\frac{\hat{w}}{\tilde{w}}(1-\tilde{w})^t(1-\bar{w})\left(\left\|\mathbf{u}^I(\hat{a})\right\|_2 + \left\|\mathbf{u}^I(a)\right\|_2\right)$ **then**
      $C \leftarrow C - \{a\}$
return $a^* \in C$

candidates that are non-dominated, and gradually prune away any candidate that is dominated by another until only one, a^*, remains.

Pseudo-code is presented in Algorithm 2. WICE first initializes $C = \mathcal{A}$ and $\omega_j^{(0)} = w_{jj}$ for all $j \in \mathcal{N}$. An iteration of WICE consists of:

1. updating estimated weights using Equation 17 for all $j \in \mathcal{N}$;
2. computing the estimated social welfare of each $a \in C$;
3. determining the maximum estimated social welfare $\hat{sw}^{(t)}$ and its corresponding alternative $\hat{a}$;
4. testing each $a \in C$ for domination, i.e.,

$$\hat{sw}^{(t)} - sw^{(t)}(a) \geq n\frac{\hat{w}}{\tilde{w}}(1-\tilde{w})^t(1-\bar{w})\left(\left\|\mathbf{u}^I(\hat{a})\right\|_2 + \left\|\mathbf{u}^I(a)\right\|_2\right);$$

$t =$	0	1	2	3	4	5	6
$\omega_1^{(t)}$	0.3	0.33	0.351	0.365	0.367	0.368	0.368
$\omega_2^{(t)}$	0.3	0.51	0.531	0.545	0.556	0.557	0.557
$\omega_3^{(t)}$	0.9	1.53	1.971	2.0151	2.046	2.067	2.070
$sw^{(t)}(a)$	0.6	0.84	0.882	0.910	0.923	0.925	0.925
$sw^{(t)}(b)$	**0.9**	**1.53**	**1.971**	**2.015**	**2.046**	**2.067**	**2.070**
$\hat{sw}^{(t)} - sw^{(t)}(a)$	—	0.69	1.089	1.105	1.123	1.142	1.145
$\hat{sw}^{(t)} - sw^{(t)}(b)$	—	0	0	0	0	0	0
$9(0.7)^t$	9	6.3	4.41	3.087	2.161	1.512	1.058

Table 3: The computation steps for the WICE algorithm over the example presented in Figure 2. The last row of the table corresponds to the right-hand side of Equation 18.

5. eliminating all dominated candidates from C.

The algorithm terminates when only one candidate (i.e., a^*) remains in C. The running time for each iteration of WICE is $O(|E| + mn)$, where $|E|$ is the number of edges.[6]

We demonstrate the steps of WICE algorithm using the earlier example in Figure 2. WICE initializes $\tilde{w} = 0.3$, $\hat{w} = 0.9$, $\bar{w} = 0.5$, $\boldsymbol{\omega}^{(0)} = (0.3, 0.3, 0.9)$ and $C = \{a, b\}$. It then iteratively computes the estimated societal weights $\boldsymbol{\omega}^{(t)}$ and social welfare as illustrated in Table 3 until $|C| = 1$. After 6 iterations, the elimination condition for candidate a is met; i.e., $\hat{sw}^{(t)} - sw^{(t)}(a) \geq n\frac{\hat{w}}{\tilde{w}}(1-\tilde{w})^t(1-\bar{w})\left(\left\|\mathbf{u}^I(\hat{a})\right\|_2 + \left\|\mathbf{u}^I(a)\right\|_2\right)$. So a is eliminated from C and the algorithm terminates with b as the winner. As with ICE, WICE does not compute social welfare exactly, but still finds the optimal alternative.

4.2.3. Comparison of ICE and WICE algorithms.

While both ICE and WICE use iterative approaches to find socially optimal alternatives, they are each useful in different contexts. If one is only interested in finding an optimal alternative, ICE may be the appropriate choice. However, in addition to finding the optimal alternative, one might be interested in learning the social weights $\boldsymbol{\omega}_g$ (e.g., for understanding the extent to which each individual influences the underlying group decision under our

[6]As discussed above for the ICE algorithm, the correctness (and termination) of WICE assumes a unique social-welfare maximizing alternative, and violation of this assumption can be addressed in the same fashion.

empathetic dynamics). In such scenarios, WICE is the appropriate choice since it computes social weights (and directly supports their approximation).

Even if the weights themselves are not required, WICE is often a more valuable method for computing optimal outcomes. Comparing the per-iteration running time of WICE and ICE iteration, $O(|E|+mn)$ vs. $O(|E|m)$, respectively, we see that WICE has better performance than ICE when $|E| \gg n$. However, we expect ICE to converge much faster (in fewer iterations) than WICE by comparing their domination conditions, Proposition 4.4 and Proposition 4.9, respectively. Specifically, the right-hand side of the inequality in Proposition 4.9 is much larger than that of Proposition 4.4, largely due to the norms on intrinsic utilities. Thus, ICE is more efficient for smaller numbers of alternatives, while WICE is well-suited for larger sets of alternatives, and specifically for problems where the alternative space cannot be enumerated (e.g., in combinatorial problems such as allocation or matching under constraints). Both methods scale reasonably well with network size, especially if the number of edges is bounded. In the next section, we compare the two algorithms empirically.

5. Empirical Results

We conduct experiments using randomly generated networks, and both randomly generated and real-world intrinsic preferences. The overall aims of the experiments are two-fold: to analyze the computational performance of our algorithms, and to contrast the decisions that result when using a standard non-empathetic approach to social-welfare maximization (i.e., using intrinsic utility only) with those that result when using the local and global empathetic models.

Experimental Setup. We assume that individual intrinsic utilities arise from an underlying preference ordering over $\mathcal{A}$.[7] In all experiments, we draw a random ordering for each agent i using either: the *impartial culture* model

[7] We use qualitative preferences in experiments for two reasons. First, we demonstrate that our models and results apply not only to utility-based group decision making, but also to decision making with qualitative preferences (under suitable scoring). Second, the lack of publicly available utility datasets has motivated us to take advantage of ranking datasets (such as the Irish voting dataset described below). We do note the recent addition of utility data to the PrefLib library, `http://www.preflib.org/` and hope to examine this in future work.

[52], in which all rankings are equally likely; or the Irish voting data set, which we explain in detail below. To draw connections to voting methods, i's utility for a is given by either the Borda or plurality score of a in its ranking r_i. Specifically, the Borda score function is defined by $s_B(a, r_i) = m - r_i(a)$ and the plurality score function is $s_P(a, r_i) = I[r_i(a) = 1]$ where $I[.]$ is an indicator function [52]. As utilities, these embody very different assumptions: Borda treats utility differences as linear, whereas plurality utility is "all or nothing." We also note that plurality has been widely-used in many group decision settings (including elections) and Borda is a useful surrogate for random utility models [53].

We generate random social networks using a *preferential attachment* model for scale-free networks [54]:[8] starting with n_0 initial nodes, we add n nodes one at a time, with a new node connected to $k \leq n_0$ existing nodes, where node i is selected as a neighbor with probability $\deg(i)/\sum_j \deg(j)$. We set $n_0 = 2$ and $k = 1$ in all experiments. We "direct" the graph by replacing each undirected edge with the two corresponding directed edges—we add a self-loop to each node with weight α; we then distribute weight $1 - \alpha$ equally to all other out-going edges. Parameter $\alpha \in (0, 1]$ represents the degree of self-interest, and $1 - \alpha$ the degree of empathy. Unless noted, all experiments have $n = 1000$ agents (nodes), $\alpha = 0.25$, and are run over 50 random preference profiles on each of 50 random networks (2500 instances).

Performance Metrics. To examine the importance of modeling empathy in social choice, we distinguish actual user preferences—referred to as the *true model*—from how preferences are modeled in a group decision-support system—namely, the *assumed model*. Specifically, we let the true and assumed models be any of our intrinsic (non-empathetic), local or global models, giving nine combinations. We are interested in the extent to which these models *disagree* in their decisions, and the loss in social welfare that results from such disagreement. If these differences are large, it indicates that, in situations where empathetic preferences exist, ignoring them by using classical preference aggregation techniques will lead to poor decisions. Specifically, we measure the percentage of *decision disagreement* (DD) (over 2500 instances for each fixed setting of other parameters) in which the true and assumed models propose different optimal decisions. We also measure the average loss

[8] This is only one of many models that can be used. Results are similar for other models.

True Model	Assumed Model		
	intrinsic	**local**	**global**
intrinsic	—	1.4(9.9)	1.1(8.0)
	—	28.4(100)	22.6(100)
local	2.9(19.3)	—	0.1(3.2)
	28.5(100)	—	1.2(86.9)
global	1.8(12.7)	0.1(2.7)	—
	22.3(100)	1.1(97.0)	—

Table 4: *Average (maximum) RSWL (1st rows) and NSWL (2nd rows): Borda, $m = 5$, Impartial Culture, $n = 1000$, $\alpha = 0.25$, 2500 runs.*

in social welfare incurred when making decisions using an assumed model that differs from the true model. Let $sw^t(\cdot)$ and $sw^a(\cdot)$ be social welfare under the true and assumed models, respectively, and a_t and a_a be the corresponding optimal options (or winners). Rather than directly comparing social welfare under various models, we define *relative social welfare loss (RSWL)* to be

$$RSWL = \frac{sw^t(a_t) - sw^t(a_a)}{sw^t(a_t)}.$$

We often report RSWL as a percentage. RSWL, by scaling differences in social welfare, helps calibrate the comparison between experiments. We can also normalize RSWL by considering the range of possible social welfare values actually attainable. Let alternative a^- have *minimum social welfare* under the true model (so it is no better than the decision under the assumed model). *Normalized social welfare loss (NSWL)* is

$$NSWL = \frac{sw^t(a_t) - sw^t(a_a)}{sw^t(a_t) - sw^t(a^-)}.$$

This offers a more realistic picture of loss caused by using an incorrect assumed utility model (by comparing it to the loss of the *worst possible decision* under the true model).

Impartial Culture. Our first experiment uses the *impartial culture model* to generate preferences—each node/agents has its preference ranking over m alternatives drawn uniformly at random from the space of all $m!$ rankings. We first consider RSWL and NSWL for all nine combinations of assumed and true utility models. We fix $m = 5$ options and use Borda scoring. Average

True Model	Assumed Model		
	intrinsic	**local**	**global**
intrinsic	—	57.76	50.48
local	58.12	—	11.72
global	50.84	11.72	—

Table 5: *Percentage decision disagreement (DD): Borda, $m = 5$, Impartial Culture, $n = 1000$, $\alpha = 0.25$, 2500 runs.*

(maximum) losses are reported in Table 4 while the decision disagreement percentage is shown in Table 5. While RSWL is relatively small on average (though maximum losses are quite large), this is largely due to the uniformity of preferences generated by impartial culture—indeed, all options have the same *expected* score. By normalizing, we obtain a more accurate picture of the loss (relative to the worst possible decision) incurred by using non-empathetic voting: average normalized loss shows that the "controllable" error is quite large, especially when comparing the "standard" intrinsic model to either of the empathetic models. Moreover, the intrinsic model chooses an incorrect (sub-optimal) alternative in over half of all instances in both cases. Interestingly, assuming either the local model or global model when the true model is the other gives reasonable results: this means that the local model offers a good first-order approximation to the global model.

Irish Voting Data. Impartial culture is often viewed as an unrealistic model of real-world preferences. For this reason, we test our methods using preferences drawn from the 2002 Irish General Election, using electoral data from the Dublin West constituency, which has 9 candidates and $29,989$ ballots of top-t form,[9] of which 3800 are complete rankings.[10] We assign full rankings, drawn randomly from the set of 3800 complete rankings to nodes in our network. Decision disagreement under both plurality and Borda scoring (Table 6) is quite high, ranging from 22–46%. Average NSWL (shown in Table 7) is not as high as with impartial culture (only in the range of 1–3%, with maximum loss of roughly 40%).

The effect of m. Figure 3 shows the average RSWL and decision disagree-

[9] A top-t ballot is a ranking over the top-t most preferred options for a voter.

[10] The original datasets were obtained from `www.dublincountyreturningofficer.com` and are currently available at PrefLib `http://www.preflib.org/`.

True Model	Assumed Model		
	intrinsic	**local**	**global**
intrinsic	—	27.3 / 46.1	22.3 / 39.0
local	28.0 / 46.3	—	5.6 / 8.6
global	22.9 / 39.3	5.5 / 8.6	—

Table 6: *Percentage decision disagreement, plurality/Borda: West Dublin dataset,* $m = 9$, $n = 1000$, $\alpha = 0.25$, *2500 runs.*

True Model	Assumed Model		
	intrinsic	**local**	**global**
intrinsic	—	1.9(34.8)	1.3(19.9)
local	2.7(39.6)	—	0.1(7.1)
global	1.6(31.7)	0.1(8.8)	—

Table 7: *Average (maximum) NSWL: 2500 runs, Plurality, West Dublin dataset,* $m = 9$, $n = 1000$, $\alpha = 0.25$.

ment (DD) for three "true vs. assumed" models as we increase the number of alternatives m using plurality scoring. We observe that average RSWL increases with m and approaches 70% when $m = 200$, while the optimal decision is rarely made. NSWL for the intrinsic model (shown in Figure 4), even at $m = 5$, averages 20–30%. With Borda scoring, the effect of m is much less pronounced because of relatively small utility differences (or smoothing) between adjacent candidates (intrinsic loss ranges from 20–30% across all values of m), but the pattern of decision disagreement is almost identical to plurality.

The effect of the scoring rule (from plurality to Borda). The experiments above use both Borda (representative of smooth scoring rules) and plurality (representative of sharp, all-or-nothing scoring rules). We now explore how RSWL changes when the scoring rule transitions from plurality to Borda. We consider the τ*−scoring rule*

$$s_\tau(a_j, r) = \tau^{r(a_j)-1}(m - r(a_j)),$$

where $r(a_j)$ represents the rank of alternative a_j in ranking r and $\tau \in [0, 1]$. Note that when $\tau = 1$, the τ-scoring rule is equivalent to Borda whereas with $\tau = 0$, the τ-scoring rule is plurality.

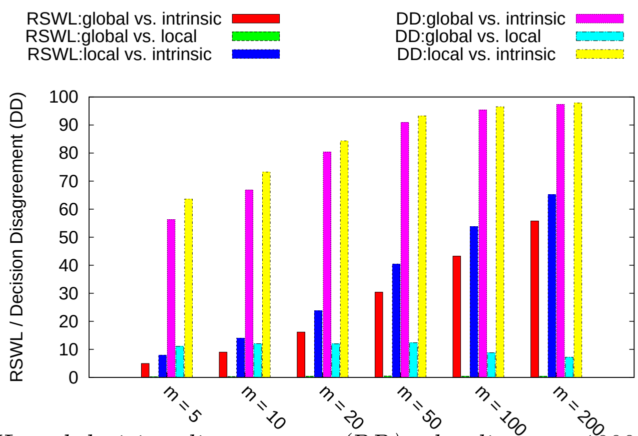


Figure 3: *RSWL and decision disagreement (DD), plurality,* $n = 1000$, $\alpha = 0.25$, *varying* m, *2500 runs.*

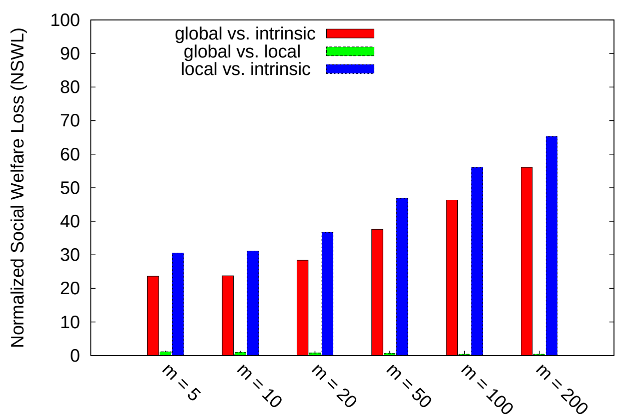


Figure 4: *Average NSWL, impartial culture, and varying the number of alternatives* m.

We set $m = 10$ and vary τ over the set $\{0, 0.2, 0.4, 0.6, 0.8, 1\}$. Figure 5 shows average (maximum, minimum) RSWL for three actual/assumed model combinations for various τ values. We observe that plurality is more susceptible to RSWL than Borda (compare $\tau = 0$ with $\tau = 1$). The change in RSWL is almost linear when moving from plurality to Borda. This implies that amongst the wide variety of scoring rules which exist between Borda and plurality, those scoring rules which are closer to plurality incur higher RSWL when compared to those are closer to Borda. These results suggest that the sharpness or smoothness of scoring rule plays a role in RSWL—the sharper the scoring rule, the higher RSWL.

The effect of self-loop weight α. In this experiment, we investigate the

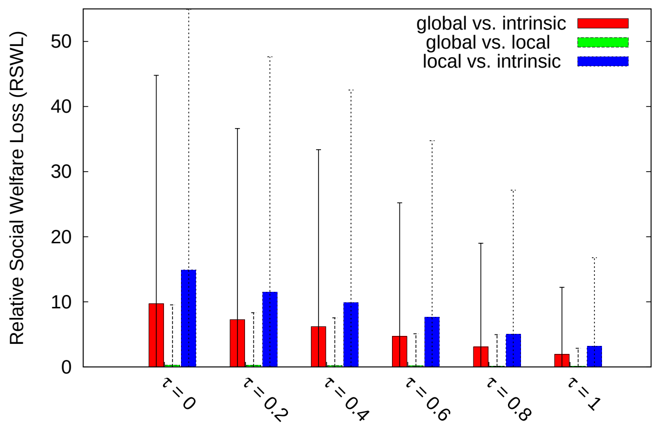


Figure 5: *Average (maximum, minimum) RSWL (2500 runs): $n = 1000$, $m = 10$, τ-scoring rule.*

α	0.05	0.1	0.25	0.5	0.75
Borda	26.0 / 58.7	25.0 / 55.8	22.2 / 53.0	15.1 / 42.5	7.3 / 28.8
Plurality	28.8 / 59.8	26.7 / 58.1	22.7 / 53.8	16.9 / 46.9	7.8 / 31.3

Table 8: *Average NSWL/decision disagreement: global vs. intrinsic, $n = 1000$, varying α, 2500 runs.*

impact of empathy in the population by controlling the self-loop weight α—higher α implies lower societal empathy. Varying α has a significant effect on NSWL and decision disagreement when true utility is global but intrinsic utility is assumed. Table 8 shows that, for both Borda and plurality, increasing α (i.e., decreasing overall degree of empathy) decreases both NWSL and DD. This is not surprising, as increasing α moves the empathetic model closer to the intrinsic model. Similar trends hold for the local model. We also examine a model in which nodes have different self-loop weights, drawing each node's α from a truncated normal distribution. As we vary the mean μ, we see a similar trend in Table 9. The results confirm that self-loops are an important factor in determining NSWL.

The effect of directionality. We now explore the effect of another aspect of social network structure on NSWL. The results above use networks with bi-directional edges (by replacing each undirected edge with two directed edges). To explore how directionality impacts NSWL, we consider networks with

μ	0.05	0.1	0.25	0.5	0.75
Borda	27.6 / 59.2	24.3 / 54.9	21.3 / 52.3	15.6 / 42.5	8.1 / 31.1
Plurality	27.2 / 58.6	24.5 / 55.5	23.3 / 54.7	16.5 / 46.1	8.0 / 31.5

Table 9: *Average NSWL/decision disagreement: global vs. intrinsic, α drawn from truncated Gaussian with mean μ and std. dev.* 0.1*,* $n = 1000$*, 2500 runs.*

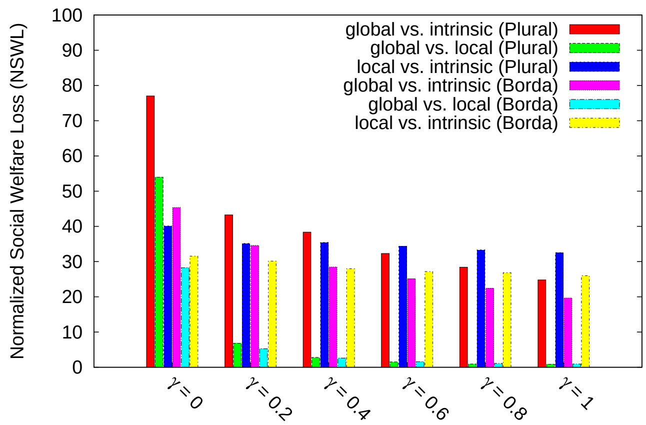


Figure 6: *Average NSWL,* $m = 10$*,* $n = 1000$*,* $\alpha = 0.25$*, varying directionality parameter* γ*, 2500 runs.*

a *hierarchical orientation*, as often found in economic (e.g., supply chain), organizational (management/employee structure), or some social networks (e.g., forms of status, following, etc.). We replace each undirected edge in the preferential attachment network with a directed edge from the "younger" node to the "older." The older node reciprocates with a directed edge to the younger with probability γ. If $\gamma = 1$, our standard bidirectional network results (as above); when $\gamma = 0$ we obtain a completely hierarchical network.

Fixing $m = 10$, Figure 6 depicts NSWL for both Borda and plurality as γ varies. Networks that are more hierarchical have higher NSWL for the global vs. intrinsic models, independent of the scoring rule, while NSWL for local vs. intrinsic is almost constant. However, plurality seems more susceptible to increasing loss due to hierarchical structure than Borda for all three combinations. Unlike earlier results, when the network is very hierarchical (e.g., $\gamma = 0$), the global and local models *do not* approximate each other well.

Our results demonstrate that the directionality of social networks has a significant impact on NSWL. Specifically, social networks with a more hierarchical structure have higher NSWL. This finding highlights the special

α	0.05	0.1	0.25	0.5	0.75
Borda	104.1	51.4	19.5	8.7	4.7
Plurality	98.7	48.6	18.6	8.3	4.6

Table 10: *Average number of iterations for ICE, $m = 10$, $n = 1000$, varying α, 2500 runs.*

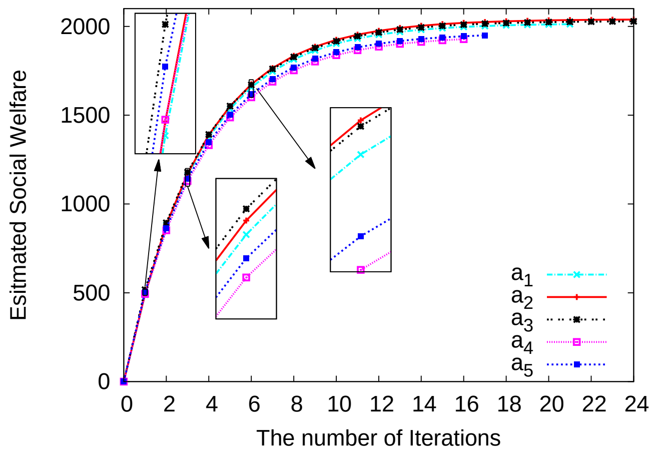


Figure 7: *Estimated social welfare vs. iterations of ICE (one sample run).*

importance of empathetic preferences for group decision making in organizations and societies with hierarchical structure.

Number of Iterations of ICE. We now assess the efficiency of our iterative algorithms. We first examine how the self-loop weight α affects the expected number of iterations required by the ICE algorithm. We fix $m = 5$, and vary α. Table 10 shows the average number of iterations for various α, for both Borda and plurality utilities. In all cases, the number of iterations is small relative to network size. ICE is quite insensitive to the scoring rule, and time-to-termination declines dramatically with increasing α, as is typical for iterative algorithms (e.g., for Markov chains). Figure 7 illustrates estimated social welfare for each alternative in one representative run ($\alpha = 0.25$, Borda scoring): this instance of ICE converges in 24 iterations, with computation time under 2 ms, despite the large number of voters. Alternative a_4 is eliminated at iteration 16, a_5 at 17, a_1 at 20, and a_2 at 24, leaving a_3 as optimal. Note that the relative order of the alternatives is unchanged after 6 iterations, suggesting that early termination may be useful as a robust means of approximation.

α	0.05	0.1	0.25	0.5	0.75
Borda	219.4	101.1	33.6	12.9	6.1
Plurality	208.1	94.6	31.5	12.1	5.7

Table 11: *Avg. number of iterations for WICE, $m = 10$, $n = 1000$, varying α, 2500 runs.*

m	5	10	20	50	100	200	500	1000
Borda	1.592	1.737	1.882	1.817	1.829	2.316	2.715	2.574
Plurality	1.691	1.923	1.930	1.957	2.227	3.105	4.209	4.330

Table 12: *The ratio of the average of running time for ICE over the average of running time for WICE (2500 common instances), $m = 10$, $n = 1000$, varying m.*

Number of Iterations of WICE. We examine how the self-loop weight α affects the expected number of iterations required by the WICE algorithm. We fix $m = 5$, and vary α. Table 11 shows the average number of iterations for various α, for both Borda and plurality utilities. The number of iterations is small relative to network size, but is almost twice that of ICE (see Table 10). WICE, similar to ICE, is quite insensitive to the scoring rule, and termination time declines dramatically with increasing α, as expected. These results demonstrate that our algorithms converge to the optimal decision in very few iterations. They also show that the less empathy present in the society, the faster convergence is.

Performance Comparison of ICE vs. WICE. The experiment above shows that WICE requires a greater number of iterations on average when compared to ICE. We here examine how the running time of ICE compares to that of WICE. We fix $\alpha = 0.25$ and $n = 1000$ but vary m. Table 12 shows the ratio of average running time for ICE over the average running time of WICE (over 2500 common instances). WICE seems to be faster than ICE despite its greater number of iterations. Given this performance, we suggest the use of WICE over ICE for applications with relatively large numbers of alternatives

Empathetic Resource Allocation. We briefly demonstrate the value of accounting for empathetic utilities in a resource-allocation problem with indivisible goods. Utilitarian social-welfare maximization for general resource-allocation problems with indivisible goods is known to be NP-complete (see the survey by Chevaleyre *et al.* [55] for an overview of resource-allocation for indivisible goods). However, in our experiments we focus on an instance of a constrained resource allocation problem which can be solved in polynomial time, since we wish to highlight and understand the importance of empathetic preferences.

We assume that there are $n = 1000$ agents and $m = 5$ distinct alternatives (or resources). There are $q_j = \frac{n}{cm}$ copies of each alternative available to be allocated to the agents, where we set constant $c = 2$ unless otherwise noted. Furthermore, we assume that each agent's ranked preferences are drawn from a Mallows ϕ-model [56]. This model is characterized by a "modal" *reference ranking* σ and a *dispersion parameter* $\phi \in [0, 1)$, with the probability of a ranking r decreasing exponentially with its Kendall-τ distance from σ: $\mathbb{P}(r|\sigma, \phi) \propto \phi^{d_\tau(r,\sigma)}$. Here, $d_\tau(r, \sigma)$ measures the number of pairwise swaps needed to transform r to σ.

An agent receives utility 1 when it is allocated its top-ranked alternative, and has zero utility for any other alternative. It is easy to show that the optimal allocation can be found by taking each alternative a_j, and greedily allocating it to the agents who ranked a_j first (until $\frac{n}{cm}$ copies have been allocated) in terms of decreasing order of the agent's societal weight. In other words, for each alternative a_j, we first find the set of agents $\mathcal{N}_j = \{i \in \mathcal{N} | r_i(a_j) = 1\}$ who ranked a_j first. We then sort the agents in $\mathcal{N}_j$ based on their societal weights ω in descending order. We start allocating a_j from the top of the sorted list until quota q_j is exhausted. After iterating over all alternatives, we arbitrarily assign the remaining alternatives to unmatched agents (because plurality scoring is used, agents are indifferent to items below their first-ranked item). Note that there are generally many possible optimal solutions given the distribution of rankings, quotas, and societal weights. Given that societal weights are pre-computed, this algorithm can be run in $O(n \log n)$ time by: first sorting individuals based on their societal weights; then iterating over the sorted list to match each individual to her most preferred item if its corresponding quota has not yet been exhausted, and to leave her unmatched otherwise; and finally going over unmatched individuals to match them with arbitrary available alternatives.

We define relative social welfare loss (RSWL) in a similar fashion to the voting scenario above:

$$RSWL = \frac{sw^t(\mathbf{x}_t) - sw^t(\mathbf{x}_a)}{sw^t(\mathbf{x}_t)},$$

where $sw^t(\cdot)$ and $sw^a(\cdot)$ are social welfare under the true and assumed models, respectively, and $\mathbf{x}_t$ and $\mathbf{x}_a$ are the corresponding optimal allocations.

Figure 8 demonstrates the average RSWL (over 2500 instances) for three "true vs. assumed" models as we increase ϕ. RSWL seems to be high for both "global vs. non-empathetic" and "local vs. non-empathetic" but it is very low for "local vs. global". RSWL decreases with ϕ, meaning that the more homogeneous the set of agents, the higher the relative social welfare loss. This is partially due to the constraints (i.e., quotas) imposed: when there are more agents with the same first-rank alternative and limited capacity, the allocation mechanisms are required to consider societal weights (due to empathy) more seriously in their allocations.[11]

We also examine the effect of resource scarcity on RSWL. We set $\phi = 0.8$ and vary c. The higher c is, the greater the scarcity of resources. Figure 9 shows the average RSWL (over 2500 instances) for various c. RSWL increases with c, suggesting that with higher scarcity, the allocation mechanism should be more cautious in assigning the resources to individuals. The general goal should be to first satisfy individuals with the higher empathetic influence (i.e., higher societal weights).

Iterative Weight Computation for Resource Allocation Finally, we examine the performance/accuracy of the weight updating scheme presented in Equation 17 in this resource-allocation setting. We assume that the true model is the global empathetic model. After each iteration, we treat the estimated weights at that iteration as an assumed model and compute RSWL using those weights relative to the optimal (converged) global model. This process allows us to monitor how RSWL evolves over iterations of our updating scheme. For this experiment, we set $c = 2$ and $\phi = 0.8$ while varying m. Table 13 shows average RSWL (over 2500 instances) for various m over 10 iterations. Average RSWL at iteration 0 increases with m. However, after only 2 iterations, average RSWL is close to zero for all m (with a maximum of almost 0.02). The average RSWL after 10 iterations (for all m) is 0.0001.

6. Concluding Remarks and Future Work

We have introduced an empathetic social choice framework in which individuals derive utility based on both their own intrinsic preferences and

[11]This pattern is not observed for consensus decision making in our experiments. In contrast, our experiments suggest that homogeneity affects RSWL in the opposite way for consensus decision making: the higher homogeneity is, the lower RSWL is.

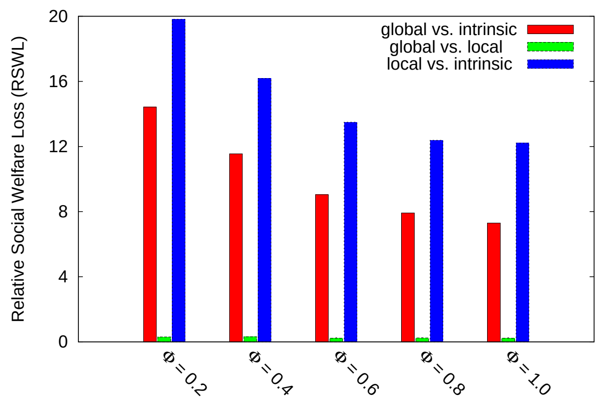


Figure 8: *RSWL for allocation problem, plurality, n=1000, m=5, q_j = 100, varying ϕ, 2500 runs.*

m/it.	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
5	7.8503	0.2459	0.0135	0.0131	0.0035	0.0045	0.0010	0.0012	0.0003	0.0003	0.0001
10	8.8486	0.2033	0.0179	0.0129	0.0040	0.0034	0.0011	0.0009	0.0003	0.0002	0.0001
20	11.5266	0.2458	0.0193	0.0203	0.0042	0.0038	0.0011	0.0009	0.0003	0.0002	0.0001
40	15.1015	0.3120	0.0187	0.0344	0.0050	0.0068	0.0014	0.0016	0.0004	0.0004	0.0001
80	17.5271	0.3521	0.0171	0.0474	0.0045	0.0092	0.0013	0.0022	0.0004	0.0006	0.0001

Table 13: *Average RSWL (estimated weights vs. global) for allocation problems as a function of the number of iterations: plurality, n=1000, $\phi = 0.8$, $q_j = \frac{n}{2m}$ for all $j \in \mathcal{A}$. Rows are indexed by m (number of alternatives. Columns are indexed by iteration number.*

empathetic preferences determined by the satisfaction of their neighbors in a social network. Using a social network to structure the degree of empathy that one agent has for another, our proposed algorithms—in both the local and global empathetic settings—allow efficient computation of optimal decisions by weighting the contribution of each agent, and have a natural interpretation as empathetic voting when scoring rules are used. Critically, individuals need only specify their intrinsic preferences (and network weights): they need not reason explicitly about the preferences of others.

We analyzed the theoretical conditions under which empathetic preferences are well-defined (i.e., converge to a fixed-point). We described conditions—*normalization*, *non-negativity*, and *positive self-loop*—under which such fixed points exist. We demonstrated that group decision making in the empathetic framework can be recast as a form of weighted social-welfare maximization. Furthermore we developed scalable algorithms for finding social welfare maximizing outcomes, taking into account the empathetic preferences of agents. We empirically demonstrated the value of accounting for empathetic preferences and the performance of our algorithms. Our theoretical and empirical

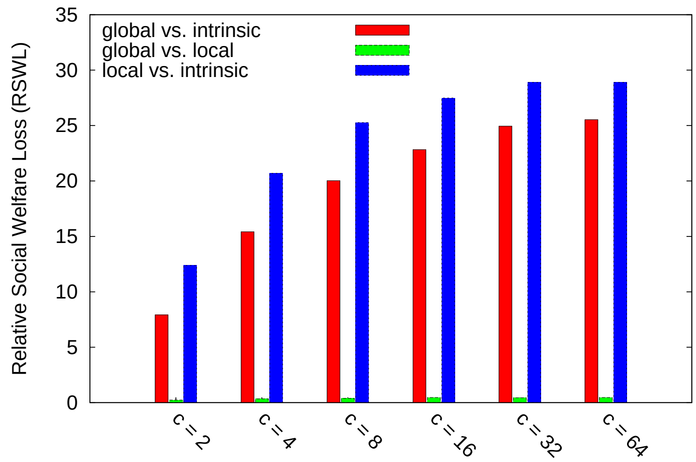


Figure 9: *RSWL for allocation problem, plurality,* $n = 1000$*,* $m = 5$*,* $\phi = 0.8$*,* $q_j = \frac{200}{c}$ *for all* $j \in \mathcal{A}$ *, varying c, 2500 runs.*

results shed light on how individual preferences become correlated due to the presence of empathy. The results also confirmed that neglecting empathetic preferences yields sub-optimal group decisions.

Our empathetic model is a starting point for the broader investigation of empathetic preferences. One can explore more realistic processes for simultaneous generation of networks and preferences that better explain preference correlation (see, e.g., the *ranking network* framework [45]). Methods to assess the prevalence of empathetic preferences, the extent to which social network structure reflects such preferences, and how they can be discovered effectively, are critical. Testing our model, and these extensions, on large data sets is, of course, important for validating the existence of empathy of this form. There are some scientifically interesting questions that one can explore in this regard. For example, to what extent and for which contexts do empathetic preferences exist in social networks? How do empathetic preferences change with the strength of social ties (e.g., are empathetic preferences stronger between couples or siblings than between classmates)?

Although our empathetic social choice framework can accommodate other social choice problems (such as assignment problems or multi-winner elections), each of these applications requires its own algorithmic developments. Of special interest is exploiting our empathetic framework for multi-winner elections. Examples of multi-winner elections with empathetic preferences are prevalent in real-world: when a city council decides to implement, say, two of a number of proposals for the use of vacant land, or when a social networking site decides to implement, say, three new functionalities selected from a slate of multiple potential new features.

As with any utility-based method, interpersonal comparison of utilities in our empathetic framework might be problematic. Although this issue can be (partially) addressed by using qualitative preferences (e.g., rankings) combined with scoring rules, an important direction is the development of ranking empathetic frameworks where agents specify tradeoffs between intrinsic and empathetic preference in a *qualitative* fashion. One can generalize our empathetic social choice framework by considering scenarios in which individuals repeatedly update their own preference rankings by aggregating their own and their neighbors' preferences (any aggregation mechanism is applicable as long as it minimizes some notion of distance between aggregated preference and local preference profiles). This *local aggregation process* might capture various psychological dynamics on social networks including empathy, confrontation, influence, imitation, etc.[12] Nonetheless, we believe that local aggregation lies at heart of each of these phenomena, since all involves an individual's preferences becoming more similar to those of (a subset of) their neighbors over time. As such, it is essential to study local aggregation by abstracting away other minor differences. Under this framework, one can study ranking preference formation and its interplay with social network structure. Of special interest is how social structure impacts the formation of correlated preference rankings on social networks, when preferences converge, and how community structures can help diversify converged preferences.

7. Acknowledgement

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Appendix A. Linear Algebra Background

In this section we provide some key definitions and results from linear algebra that are used in the proofs for this paper.

Definition (Spectrum $\sigma(\mathbf{A})$). The set of eigenvalues of an $n \times n$ matrix $\mathbf{A}$ is called its *spectrum* $\sigma(\mathbf{A})$.

Definition (Spectral Radius $\rho(\mathbf{A})$). Let $\mathbf{A}$ be an $n \times n$ matrix with real or complex eigenvalues $\sigma(\mathbf{A})$. Then the *spectral radius* of $\mathbf{A}$ is

$$\rho(\mathbf{A}) = \max_{\lambda \in \sigma(\mathbf{A})} |\lambda|$$

.

Definition (M-matrix). A matrix $\mathbf{A}$ in the form of $\mathbf{A} = s\mathbf{I} - \mathbf{B}$ is an *M-matrix* if $s \geq \rho(\mathbf{B})$ and $\mathbf{B} \geq \mathbf{0}$.

Proposition A1 (Nonsingular M-matrix [59]). *If $s > \rho(\mathbf{B})$ in an M-matrix $\mathbf{A} = s\mathbf{I} - \mathbf{B}$, then $\mathbf{A}$ is nonsingular and $\mathbf{A}^{-1} \geq \mathbf{0}$*

Note that an M-matrix can be either singular or nonsingular. Therefore, the condition $s > \rho(\mathbf{B})$ in Proposition A1 is necessary to guarantee the nonsingularity of an M-matrix.[13]

Theorem A2 (Gerschgorin Circles [59])**.** *The eigenvalues of matrix* $\mathbf{A} \in \mathcal{C}^{n\times n}$ *are contained in* $\cup_{i=1}^{n}\mathcal{G}_i$*, where* $\mathcal{G}_i$ *is the Gerschgorin circle defined by:*

$$\mathcal{G}_i = \left\{c \in \mathcal{C} \middle| |c - a_{ii}| \leq R_i\right\} \text{ where } R_i = \sum_{\substack{1\leq j\leq n \\ j\neq i}} |a_{ij}|$$

We exploit induced matrix norms in our analysis of convergence rate of our iterative method for fixed-point utilities. For a given vector norm $\|.\|$, the induced norm for $n \times m$ matrix $\mathbf{A} \in \mathcal{C}^{n\times m}$ is:

$$\begin{aligned}\|\mathbf{A}\| &= \max\left\{\|\mathbf{A}\mathbf{x}\| : \mathbf{x} \in \mathcal{C}^m \text{ and } \|\mathbf{x}\| = 1\right\} \\ &= \max\left\{\frac{\|\mathbf{A}\mathbf{x}\|}{\|\mathbf{x}\|} : \mathbf{x} \in \mathcal{C}^m \text{ and } \mathbf{x} \neq \mathbf{0}\right\}\end{aligned}$$

We here focus on the p-norm $\|.\|_p$ which is induced by the p-norm in vector spaces. More precisely, the p-norm of matrix $\mathbf{A}$ is

$$\|\mathbf{A}\|_p = \max_{\mathbf{x}\neq\mathbf{0}} \frac{\|\mathbf{A}\mathbf{x}\|_p}{\|\mathbf{x}\|_p}$$

where the p-norm $\|\mathbf{x}\|_p$ of vector $\mathbf{x} \in \mathcal{C}^n$ is:

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$$

p-norms have several important properties: (1) they are *submultiplicative*: $\|\mathbf{A}\mathbf{B}\|_p \leq \|\mathbf{A}\|_p\|\mathbf{B}\|_p$. A consequence of this consistency property is that, for any square matrix $\mathbf{A}$, $\|\mathbf{A}^k\|_p \leq \|\mathbf{A}\|_p^k$. (2) By definition, they are *compatible*: $\|\mathbf{A}\mathbf{x}\|_p \leq \|\mathbf{A}\|_p\|\mathbf{x}\|_p$ where $\mathbf{A} \in \mathcal{C}^{n\times m}$ and $\mathbf{x} \in \mathcal{C}^m$.

[13] In some references (e.g., [59]), an M-matrix is defined with $s > \rho(\mathbf{B})$. By this definition, an M-matrix is nonsingular.

For the cases where $p = 1$ or $p = \infty$, the matrix p-norm can be computed easily. The 1-norm for matrix $\mathbf{A}$ is simply the maximum absolute column sum of $\mathbf{A}$:

$$\|\mathbf{A}\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{m} |a_{ij}| \tag{A.1}$$

The ∞-norm for matrix $\mathbf{A}$ is simply the maximum absolute row sum of $\mathbf{A}$:

$$\|\mathbf{A}\|_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^{n} |a_{ij}|. \tag{A.2}$$

We review the Jacobi iterative method and its convergence criteria and rate. Iterative methods offer practical advantages for solving linear systems [51]. A linear system is formally defined as follows: Given an $n \times n$ real-valued matrix $\mathbf{A}$ and a real n-vector $\mathbf{b}$, the problem is to find n-vector $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{A}\mathbf{x} = \mathbf{b}$.

The Jacobi method [51] is an iterative method for solving linear systems. Consider this decomposition $\mathbf{A} = \mathbf{\Lambda} - \mathbf{E} - \mathbf{F}$ where $\mathbf{\Lambda}$ is the diagonal matrix of $\mathbf{A}$, $\mathbf{E}$ is the strictly lower triangular matrix of $-\mathbf{A}$, and $\mathbf{F}$ is the strictly upper triangular matrix of $-\mathbf{A}$. Note that we assume that the diagonal entries of $\mathbf{A}$ are all non-zero (this corresponds to our positive self-loop assumption below). Each iteration of the Jacobi method takes the form of:

$$\mathbf{x}^{(t+1)} = \mathbf{\Lambda}^{-1}(\mathbf{E} + \mathbf{F})\mathbf{x}^{(t)} + \mathbf{\Lambda}^{-1}\mathbf{b} \tag{A.3}$$

Theorem A3 (Convergence of Iterative Methods [51])**.** *Let an iterative method take the form of* $\mathbf{x}_{t+1} = \mathbf{G}\mathbf{x}_t + \mathbf{f}$ *where* $\mathbf{G}$ *is an* $n \times n$ *iteration matrix and* $\mathbf{f}$ *is an* n*-vector. It converges if and only if* $\rho(\mathbf{G}) < 1$*.*

Corollary A1 (Jacobi Convergence)**.** *The Jacobi iterative method converges to the solution of linear system* $\mathbf{A}\mathbf{x} = \mathbf{b}$ *if* $\rho(\mathbf{G}) < 1$ *where* $\mathbf{G} = \mathbf{\Lambda}^{-1}(\mathbf{E} + \mathbf{F})$*.*

Proof The proof of convergence is trivial and immediately follows form the Theorem A3 by letting $\mathbf{G} = \mathbf{\Lambda}^{-1}(\mathbf{E} + \mathbf{F})$ and $\mathbf{f} = \mathbf{\Lambda}^{-1}\mathbf{b}$. Now, we prove that the Jacobi method converges to the solution of the linear system. Since it converges, let $\mathbf{x}^* = \lim_{t\to\infty} \mathbf{x}^{(t)}$. From Equation A.3, we have:

$$\begin{aligned}
&\lim_{t\to\infty} \mathbf{x}^{(t+1)} = \lim_{t\to\infty} \mathbf{\Lambda}^{-1}(\mathbf{E}+\mathbf{F})\mathbf{x}^{(t)} + \mathbf{\Lambda}^{-1}\mathbf{b} \\
\Longrightarrow\ &\lim_{t\to\infty} \mathbf{x}^{(t+1)} = \mathbf{\Lambda}^{-1}(\mathbf{E}+\mathbf{F})\left(\lim_{t\to\infty} \mathbf{x}^{(t)}\right) + \mathbf{\Lambda}^{-1}\mathbf{b} \\
\Longrightarrow\ &\mathbf{x}^* = \mathbf{\Lambda}^{-1}(\mathbf{E}+\mathbf{F})\mathbf{x}^* + \mathbf{\Lambda}^{-1}\mathbf{b} \\
\Longrightarrow\ &\mathbf{\Lambda}\mathbf{x}^* = (\mathbf{E}+\mathbf{F})\mathbf{x}^* + \mathbf{b} \\
\Longrightarrow\ &(\mathbf{\Lambda}-\mathbf{E}-\mathbf{F})\mathbf{x}^* = \mathbf{b} \\
\Longrightarrow\ &\mathbf{A}\mathbf{x}^* = \mathbf{b}
\end{aligned}$$

So $\mathbf{x}^*$ is the solution of the linear system. ∎

Appendix B. Proofs

Appendix B.1. Proof of Proposition 3.1

In this section we provide the proof for Proposition 3.1 and Corollary 3.1. Recall that if we are given a weighted graph $G = (\mathcal{N}, E)$ then the adjacency matrix is $\mathbf{W} = [w_{ij}]$ and $\mathbf{D}$ is the corresponding diagonal matrix where $d_{jj} = w_{jj}$.

Lemma B1. *Given $G = (\mathcal{N}, E)$ with weight matrix $\mathbf{W}$, if non-negativity holds, $\mathbf{W}$ is normalized and agents have positive self-loops, then $\rho(\mathbf{B}) < 1$ where $\mathbf{B} = \mathbf{W} - \mathbf{D}$.*

Proof By the definition of $\mathbf{W}$ and $\mathbf{D}$, it can be seen that $\mathbf{B} = \mathbf{W} - \mathbf{D}$ is a matrix with $b_{ii} = 0$ and $b_{ij} = w_{ij}$ for all $i, j \in \mathcal{N}$ and $i \neq j$. Using the Gerschgorin Circle Theorem (Theorem A2), we have $\sigma(\mathbf{B}) \subset \cup_{i=1}^{n} \mathcal{G}_i$ where

$$\mathcal{G}_i = \left\{ c \in \mathcal{C} \middle| |c - b_{ii}| \leq R_i \right\} \text{ and } R_i = \sum_{\substack{1\leq j\leq n \\ j\neq i}} |b_{ij}|.$$

As $b_{ii} = 0$ and $b_{ij} = w_{ij}$ for $i \neq j$, we have:

$$\mathcal{G}_i = \left\{ c \in \mathcal{C} \middle| |c| \leq R_i \right\} \text{ where } R_i = \sum_{\substack{1\leq j\leq n \\ j\neq i}} |w_{ij}|.$$

Note that each $\mathcal{G}_i$ is a closed disk in $\mathcal{C}$ which is centered at 0. So $\cup_{i=1}^{n} \mathcal{G}_i$ is the union of closed disks of various radii but the same center 0. Since the number of these disks is finite, we can cover all these closed disks with a closed covering disk defined by $\{c \in \mathcal{C} \big| |c| \leq R_{max}\}$ where $R_{max} = \max_{i=1}^{n} R_i$. Without loss of generality, let $l = \arg\max_i R_i$. So, we have

$$\sigma(\mathbf{B}) \subset \cup_{i=1}^{n} \{c \in \mathcal{C} \big| |c| \leq R_i\} = \{c \in \mathcal{C} \big| |c| \leq R_l\}$$

From this, it follows that:

$$|\lambda| \leq R_l, \forall \lambda \in \sigma(\mathbf{B}) \implies \max_{\lambda \in \sigma(\mathbf{B})} |\lambda| \leq R_l \implies \rho(\mathbf{B}) \leq R_l.$$

Using $R_l = \sum_{j \neq l} |w_{lj}|$ and the normalization assumption $\sum_j w_{lj} = 1$, we have $\rho(\mathbf{B}) \leq 1 - w_{ll}$. Since $w_{ll} > 0$ by self-loop positivity, we have $\rho(\mathbf{B}) \leq 1 - w_{ll} < 1$. ∎

We now restate Proposition 3.1.

Proposition 3.1. *Given $G = (\mathcal{N}, E)$ with weight matrix $\mathbf{W}$, if non-negativity holds, $\mathbf{W}$ is normalized and agents have positive self-loops, then there is a unique solution to*

$$\mathbf{u}(a) = (\mathbf{W} - \mathbf{D})\mathbf{u}(a) + \mathbf{D}\mathbf{u}^I(a).$$

In particular,

$$\mathbf{u}(a) = (\mathbf{I} - \mathbf{W} + \mathbf{D})^{-1}\mathbf{D}\mathbf{u}^I(a).$$

Proof Using Equation 8, we can write:

$$\begin{aligned} &\mathbf{u}(a) = (\mathbf{W} - \mathbf{D})\mathbf{u}(a) + \mathbf{D}\mathbf{u}^I(a) \\ &\implies \mathbf{u}(a) - (\mathbf{W} - \mathbf{D})\mathbf{u}(a) = \mathbf{D}\mathbf{u}^I(a) \\ &\implies (\mathbf{I} - (\mathbf{W} - \mathbf{D}))\mathbf{u}(a) = \mathbf{D}\mathbf{u}^I(a) \end{aligned}$$

So it is sufficient to show that $(\mathbf{I} - (\mathbf{W} - \mathbf{D}))^{-1}$ exists to prove that $\mathbf{u}(a) = (\mathbf{I} - \mathbf{W} + \mathbf{D})^{-1}\mathbf{D}\mathbf{u}(a)$ exists and is unique. We need to show that $\mathbf{I} - (\mathbf{W} - \mathbf{D})$ is nonsingular to guarantee the existence of $(\mathbf{I} - (\mathbf{W} - \mathbf{D}))^{-1}$.

Let $\mathbf{B} = \mathbf{W} - \mathbf{D}$. By definitions of $\mathbf{W}$ and $\mathbf{D}$, the matrix $\mathbf{B}$ has $b_{ii} = 0$ and $b_{ij} = w_{ij}$ for all $i, j \in \mathcal{N}$ and $i \neq j$. By the non-negativity assumption, we have $w_{ij} \geq 0$, so $\mathbf{B} \geq \mathbf{0}$. By setting $s = 1$, $(\mathbf{I} - (\mathbf{W} - \mathbf{D})) = (s\mathbf{I} - \mathbf{B})$ which is an M-matrix. Using Lemma B1, we have $\rho(\mathbf{B}) < 1$. Since $s = 1$, then $\rho(\mathbf{B}) < s$. By Proposition A1, it follows that $(\mathbf{I} - (\mathbf{W} - \mathbf{D}))$ is nonsingular and $(\mathbf{I} - (\mathbf{W} - \mathbf{D}))^{-1} \geq \mathbf{0}$. ∎

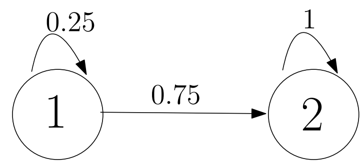

Figure B.10: The social network of our counter example for demonstrating that the empathetic model is not subsumed by the metric labeling model.

Corollary 3.1. *Given $G = (\mathcal{N}, E)$ with weight matrix $\mathbf{W}$, the (global) social welfare of alternative a is given by*

$$sw_g(\mathbf{u}, a) = \omega_g^T \mathbf{u}^I \tag{B.1}$$

where

$$\omega_g^T = \mathbf{e}^T(\mathbf{I} - \mathbf{W} + \mathbf{D})^{-1}\mathbf{D}.$$

Proof From the definition of social welfare and Proposition 3.1, it follows

$$sw_g(a, \mathbf{u}^I) = \mathbf{e}^\top(\mathbf{I} - \mathbf{W} + \mathbf{D})^{-1}\mathbf{D}\mathbf{u}^I(a).$$

By setting $\boldsymbol{\omega}_g^\top = \mathbf{e}^\top(\mathbf{I} - \mathbf{W} + \mathbf{D})^{-1}\mathbf{D}$, we have $sw_g(a, \mathbf{u}^I) = \boldsymbol{\omega}_g^\top \mathbf{u}^I$. ∎

Appendix B.2. Proof of Proposition3.2

Proposition 3.2. *The empathetic model is not subsumed by the metric labeling model.*

Proof By a counter example, we show that the empathetic model is not subsumed by the metric labeling model.

First, we note that Equation 12 can be written as:

$$u_i(\mathbf{x}) = w_{ii}u_i^I(x^i) + \beta_0 \sum_j w_{ij} - \beta_1 \sum_j w_{ij} d(x^i, x^j).$$

Now consider the simple social network depicted in Figure B.10. We assume there are two items a and b and the set of feasible allocations is $\mathcal{A} = (a, a), (a, b), (b, a), (b, b)$. Let agents 1 and 2 have the following intrinsic utilities over a and b: $u_1^I(a) = 1$, $u_2^I(a) = 0$, $u_1^I(b) = 0$, $u_2^I(b) = 1$. We now focus on agent 1's overall utility $u_1(.|x^2 = b)$ assuming agent 2 is assigned to item b (i.e., $x^2 = b$).

Let $u_1^L(.|x^2 = b)$ be the overall utility of individual 1 given $x^2 = b$ under the labeling framework. Then, we have $u_1^L(x^1|x^2 = b) = 0.25 \times u^I(x^1) +$

$0.75\beta_0 - 0.75 \times \beta_1 d(x^1, b)$. Given this, $u_1^L(a|x^2 = b) = 0.25 + 0.75\beta_0 - 0.75\beta_1 \times d(a, b)$. Similarly, $u_1^L(b|x^2 = b) = 0.75\beta_0 - 0.75\beta_1 \times d(b, b)$. As $d(b, b) = 0$ for a valid distance function due to identity of indiscernibles, we have $u_1^L(b|x^2 = b) = 0.75\beta_0$.

Let $u_1^E(.|x^2 = b)$ be the overall utility of individual 1 given $x^2 = b$ under empathetic framework. Then we have $u_1^E(a|x^2 = b) = 0.25 \times u_1^I(a) + 0.75 \times u_2^E(b|x^2 = b)$. As $u_2^E(b|x^2 = b) = 1.0 \times u_2^I(b)$ and given the intrinsic utilities above, we have $u_1^E(a|x^2 = b) = 0.25 \times 1 + 0.75 \times 1 = 1$. Similarly, $u_1^E(b|x^2 = b) = 0.75$.

We need only check whether there are valid values for β_0, β_1 and $d(., .)$ which map the empathetic model to the labeling model. By letting $u_1^L(b|x^2 = b) = u_1^E(b|x^2 = b)$, we have $\beta_0 = 1$. Letting $u_1^L(a|x^2 = b) = u_1^E(a|x^2 = b)$ and $\beta_0 = 1$, we have $1 - 0.75\beta_1 d(a, b) = 1$, thus $d(a, b) = 0$ or $\beta_1 = 0$. However, $\beta_1 > 0$ by assumption and $d(a, b)$ cannot be zero for a valid distance function. As a consequence, the empathetic model is not subsumed by the metric labeling model. ∎

Appendix B.3. Proofs of Results from Section 4.2.1

Theorem 4.1. *Given graph $G = (\mathcal{N}, E)$ with weight matrix $\mathbf{W}$, alternative set $\mathcal{A}$ and intrinsic utilities $\mathbf{u}^I$, consider the following iteration:*

$$\mathbf{u}^{(t+1)}(a) = (\mathbf{W} - \mathbf{D})\mathbf{u}^{(t)}(a) + \mathbf{D}\mathbf{u}^I(a). \tag{B.2}$$

Assuming non-negativity, normalization, and positive self-loop, this method converges to $\mathbf{u}(a)$, the solution to Equation 8.

Proof of Theorem 4.1 From Equation 8, we observe that $\mathbf{u}(a)$ is the solution of the linear system $\mathbf{A}\mathbf{u}(a) = \mathbf{b}$ with $\mathbf{A} = \mathbf{I} - (\mathbf{W} - \mathbf{D})$ and $\mathbf{b} = \mathbf{D}\mathbf{u}^I(a)$. The Jacobi method is:

$$\mathbf{u}(a)^{(t+1)} = \mathbf{\Lambda}^{-1}(\mathbf{E} + \mathbf{F})\mathbf{u}(a)^{(t)} + \mathbf{\Lambda}^{-1}\mathbf{b}$$

Since $\mathbf{A} = \mathbf{I} - (\mathbf{W} - \mathbf{D})$, we have that $\mathbf{\Lambda} = \mathbf{I}$ and $\mathbf{E} + \mathbf{F} = \mathbf{W} - \mathbf{D}$. As $\mathbf{b} = \mathbf{D}\mathbf{u}^I(a)$, we have:

$$\begin{aligned} \mathbf{u}(a)^{(t+1)} &= \mathbf{I}^{-1}(\mathbf{W} - \mathbf{D})\mathbf{u}(a)^{(t)} + \mathbf{I}^{-1}\mathbf{D}\mathbf{u}^I(a) \\ \Longrightarrow \mathbf{u}(a)^{(t+1)} &= (\mathbf{W} - \mathbf{D})\mathbf{u}(a)^{(t)} + \mathbf{D}\mathbf{u}^I(a) \end{aligned}$$

From Lemma B1, we have $\rho(\mathbf{W} - \mathbf{D}) < 1$. Then, using Corollary A1, we know that $\mathbf{u}^{(t+1)}(a) = (\mathbf{W} - \mathbf{D})\mathbf{u}^{(t)}(a) + \mathbf{D}\mathbf{u}^I(a)$ converges to $\mathbf{u}(a)$ which is the solution to the linear system. ∎

Theorem 4.2. *In the iterative scheme above,*

$$\left\|\mathbf{u}(a)-\mathbf{u}^{(t)}(a)\right\|_\infty \leq (1-\tilde{w})^t \left\|\mathbf{u}(a)-\mathbf{u}^{(0)}(a)\right\|_\infty,$$

where $\tilde{w}=\min_{1\leq i\leq n} w_{ii}$.

Proof of Theorem 4.2 Using Equation 8 and $\mathbf{u}^{(t)}(a)=(\mathbf{W}-\mathbf{D})\mathbf{u}^{(t-1)}(a)+\mathbf{D}\mathbf{u}^I(a)$, we can write $\mathbf{u}(a)-\mathbf{u}^{(t)}(a)=(\mathbf{W}-\mathbf{D})(\mathbf{u}(a)-\mathbf{u}^{(t-1)}(a))$. By induction on t, we have $\mathbf{u}(a)-\mathbf{u}^{(t)}(a)=(\mathbf{W}-\mathbf{D})^t\left(\mathbf{u}(a)-\mathbf{u}^{(0)}(a)\right)$. Thus, we have

$$\begin{aligned}
&\left\|\mathbf{u}(a)-\mathbf{u}^{(t)}(a)\right\|_\infty = \left\|(\mathbf{W}-\mathbf{D})^t\left(\mathbf{u}(a)-\mathbf{u}^{(0)}(a)\right)\right\|_\infty\\
&\leq \left\|(\mathbf{W}-\mathbf{D})^t\right\|_\infty \left\|\mathbf{u}(a)-\mathbf{u}^{(0)}(a)\right\|_\infty \text{(by compatibility)}\\
&\leq \left\|\mathbf{W}-\mathbf{D}\right\|_\infty^t \left\|\mathbf{u}(a)-\mathbf{u}^{(0)}(a)\right\|_\infty \text{(by consistency)}\\
&= \left(\max_{1\leq i\leq n}\sum_{j=1}^{n}|w_{ij}-d_{ij}|\right)^t \left\|\mathbf{u}(a)-\mathbf{u}^{(0)}(a)\right\|_\infty (\infty\text{-norm})\\
&= \left(\max_{1\leq i\leq n}\sum_{\substack{j=1\\ j\neq i}}^{n}|w_{ij}|\right)^t \left\|\mathbf{u}(a)-\mathbf{u}^{(0)}(a)\right\|_\infty \text{(by defn. of }\mathbf{D}\text{)}\\
&= \left(\max_{1\leq i\leq n}\sum_{\substack{j=1\\ j\neq i}}^{n}w_{ij}\right)^t \left\|\mathbf{u}(a)-\mathbf{u}^{(0)}(a)\right\|_\infty \text{(by non-negativity)}\\
&= \left(1-\min_{1\leq i\leq n} w_{ii}\right)^t \left\|\mathbf{u}(a)-\mathbf{u}^{(0)}(a)\right\|_\infty \text{(by normalization)}
\end{aligned}$$

Letting $\tilde{w}=\min_{1\leq i\leq n} w_{ii}$, we have shown that

$$\left\|\mathbf{u}(a)-\mathbf{u}^{(t)}(a)\right\|_\infty \leq (1-\tilde{w})^t \left\|\mathbf{u}(a)-\mathbf{u}^{(0)}(a)\right\|_\infty. \quad \blacksquare$$

Lemma B2. *Assume non-negativity and normalization, consider the iterative updating scheme:* $\mathbf{u}^{(t)}(a)=(\mathbf{W}-\mathbf{D})\mathbf{u}^{(t-1)}(a)+\mathbf{D}\mathbf{u}^I(a)$*. If* $\forall i\in\mathcal{N}$, $u_i^I(a)\in[c,d]$ *and* $u_i^{(0)}(a)\in[c,d]$*, then* $u_i(a)^{(t)}\in[c,d]$, $\forall i\in\mathcal{N}$ *and* $\forall t\in\mathbb{N}$*. Moreover, we have* $u_i(a)\in[c,d]$, $\forall i\in\mathcal{N}$*.*

Proof We first prove the first part of the lemma by induction on t. The base case is $t = 0$ for which it is given that $u_i^{(0)}(a) \in [c, d]$, $\forall i \in \mathcal{N}$. The induction hypothesis is that $u_i^{(t)}(a) \in [c, d]$ for all $\forall i \in \mathcal{N}$. There are two useful inequalities which follow immediately from the induction hypothesis: $\max_{i \in \mathcal{N}} u_i^{(t)}(a) \leq d$ and $\min_{i \in \mathcal{N}} u_i^{(t)}(a) \geq c$. We can write the updating scheme for each individual $i \in \mathcal{N}$ and the alternative $a \in \mathcal{A}$ as follows:

$$u_i^{(t+1)}(a) = w_{ii} u_i^I(a) + \sum_{k \neq i} w_{ik} u_k^{(t)}(a) \tag{B.3}$$

where $u_i^{(t)}(a)$ denotes the utility of individual i for alternative a after t iterations. Using this equation and the two inequalities mentioned above, we will first show the upper bound on d and then the lower bound on c for $u_i^{(t+1)}(a)$, $\forall i \in \mathcal{N}$, and fixed a. For the upper bound, we can write the following:

$$\begin{aligned}
u_i^{(t+1)}(a) \leq & \max_{j \in \mathcal{N}} \left\{ u_j^{(t+1)}(a) \right\} = \max_{j \in \mathcal{N}} \left\{ w_{jj} u_j^I(a) + \sum_{k \neq j} w_{jk} u_k^{(t)}(a) \right\} \\
\leq & w_{jj} \max_{j \in \mathcal{N}} \left\{ u_j^I(a) \right\} + \sum_{k \neq j} w_{jk} \max_{j \in \mathcal{N}} \left\{ u_k^{(t)}(a) \right\} \\
\leq & w_{jj} d + \sum_{k \neq j} w_{jk} d = d \sum_{k} w_{jk} = d.
\end{aligned}$$

Similarly, for the lower bound:

$$\begin{aligned}
u_i^{(t+1)}(a) \geq & \min_{j \in \mathcal{N}} \left\{ u_j^{(t+1)}(a) \right\} = \min_{j \in \mathcal{N}} \left\{ w_{jj} u_j^I(a) + \sum_{k \neq j} w_{jk} u_k^{(t)}(a) \right\} \\
\geq & w_{jj} \min_{j \in \mathcal{N}} \left\{ u_j^I(a) \right\} + \sum_{k \neq j} w_{jk} \min_{j \in \mathcal{N}} \left\{ u_k^{(t)}(a) \right\} \\
\geq & w_{jj} c + \sum_{k \neq j} w_{ik} c = c \sum_{k} w_{jk} = c.
\end{aligned}$$

So we have shown that $c \leq u_i^{(t+1)}(a) \leq d$, $\forall i \in \mathcal{N}$ and $a \in \mathcal{A}$, so proving the first part of the lemma.

Now, we will prove the second part of lemma by showing that $u_i(a) \in [c, d]$, $\forall i \in \mathcal{N}$ and $a \in \mathcal{A}$. Fix an arbitrary $i \in \mathcal{N}$. The sequence $u_i^{(t)}(a)$

with $t = 0, 1, 2, \ldots$ is a convergent sequence which converges to $u_i(a) = lim_{t\to\infty} u_i^{(t)}(a)$ (based on Theorem 4.1). Note that, from the first part of this lemma, we have $u_i^{(t)}(a) \in [c, d]$ for any $t \in \mathbb{N} \cup \{0\}$. So we can see $u_i^{(t)}(a)$ is a convergent sequence on the closed set $[c, d]$. As $[c, d]$ is closed, the limit point of $u_i^{(t)}(a)$ sequence which is $u_i(a)$ must belong to $[c, d]$. As i and a are chosen arbitrarily, we have $u_i(a) \in [c, d]$. ∎

Theorem 4.3. *Assume $u_j^I(a), u_j^{(0)}(a) \in [c, d]$, for all j. Then*

$$\left|sw(a) - sw^{(t)}(a)\right| \leq n(d-c)\left(1-\tilde{w}\right)^t,$$

for all t, under the conditions above, where $\tilde{w} = \min_{1\leq i\leq n} w_{ii}$.

Proof of Theorem 4.3 Let $\tilde{w} = \min_{1\leq i\leq n} w_{ii}$. Using Theorem 4.2, we can write

$$\begin{aligned}
&\left\|\mathbf{u}(a) - \mathbf{u}^{(t)}(a)\right\|_\infty \leq (1-\tilde{w})^t \left\|\mathbf{u}(a) - \mathbf{u}^{(0)}(a)\right\|_\infty \implies \\
&\left|u_i(a) - u_i^{(t)}(a)\right| \leq \max_i \left|u_i(a) - u_i^{(t)}(a)\right| \\
&\leq (1-\tilde{w})^t \left\|\mathbf{u}(a) - \mathbf{u}^{(0)}(a)\right\|_\infty \implies \\
&\sum_{i=1}^n \left|u_i(a) - u_i^{(t)}(a)\right| \leq n\,(1-\tilde{w})^t \left\|\mathbf{u}(a) - \mathbf{u}^{(0)}(a)\right\|_\infty .
\end{aligned} \tag{B.4}$$

By Lemma B2, we know that $u_i(a) \in [c, d]$. Based on this and the assumption that $u_i^{(0)}(a) \in [c, d]$, it follows that $|u_i(a) - u_i^{(0)}(a)| \leq d - c$. So we can continue Inequality (B.4) as follows:

$$\begin{aligned}
\sum_{i=1}^n \left|u_i(a) - u^{(t)}(a)\right| &\leq n\,(1-\tilde{w})^t \left\|\mathbf{u}(a) - \mathbf{u}^{(0)}(a)\right\|_\infty \\
&\leq n(d-c)\,(1-\tilde{w})^t .
\end{aligned} \tag{B.5}$$

By Lemma B2, we know that $u_i(a) \in [c, d]$ and $u_i^{(t)}(a) \in [c, d]$, $\forall t \in \mathbb{N} \cup \{0\}$. By the triangle inequality, we have:

$$\begin{aligned}
\sum_{i=1}^n \left|u_i(a) - u_i^{(t)}(a)\right| &\geq \left|\sum_{i=1}^n \left(u_i(a) - u_i^{(t)}(a)\right)\right| \\
&= \left|\sum_{i=1}^n u_i(a) - \sum_{i=1}^n u_i^{(t)}(a)\right| \\
&= \left|sw(a) - sw^{(t)}(a)\right| .
\end{aligned} \tag{B.6}$$

By Inequalities (B.5) and (B.6), we conclude that

$$\left|sw(a) - sw^{(t)}(a)\right| \leq n(d-c)\,(1-\tilde{w})^t\,,$$

where $\tilde{w} = \min_{1 \leq i \leq n} w_{ii}$. ∎

Proposition 4.4. *If $sw^{(t)}(b) - sw^{(t)}(a) \geq 2n(d-c)\,(1-\tilde{w})^t$ then $sw(b) > sw(a)$.*

Proof of Proposition 4.4 Using the triangle inequality and the inequality presented in Theorem 4.3, we can write:

$$\begin{aligned}
&sw^{(t)}(b) - sw^{(t)}(a) \\
&= sw^{(t)}(b) - sw(b) + sw(b) - sw^{(t)}(a) + sw(a) - sw(a) \\
&\leq |sw^{(t)}(b) - sw(b)| + sw(b) + |sw(a) - sw^{(t)}(a)| - sw(a) \\
&\leq n(d-c)(1-\tilde{w})^t + sw(b) + n(d-c)(1-\tilde{w})^t - sw(a) \\
&= 2n(d-c)(1-\tilde{w})^t + sw(b) - sw(a).
\end{aligned}$$

Using this and $sw^{(t)}(b) - sw^{(t)}(a) \geq 2n(d-c)(1-\tilde{w})^t$, we have $2n(d-c)(1-\tilde{w})^t \leq 2n(d-c)(1-\tilde{w})^t + sw(b) - sw(a)$. This implies $sw(b) \geq sw(a)$. ∎

Appendix B.4. Proofs of Results from Section 4.2.2

Theorem 4.5. *$\boldsymbol{\omega}_g$ is the unique solution to the linear system of $\mathbf{A}\boldsymbol{\omega}_g = \mathbf{e}$ where $\mathbf{A} = (\mathbf{I} - \mathbf{W}^\top + \mathbf{D})\mathbf{D}^{-1}$.*

Proof of Theorem 4.5 The proof is trivial. From Corollary 3.1, we have

$$\begin{aligned}
\boldsymbol{\omega}_g^\top &= \mathbf{e}^\top(\mathbf{I} - \mathbf{W} + \mathbf{D})^{-1}\mathbf{D} \\
\Longrightarrow \boldsymbol{\omega}_g &= \mathbf{D}^\top\left((\mathbf{I} - \mathbf{W} + \mathbf{D})^{-1}\right)^\top \mathbf{e} \\
\Longrightarrow \boldsymbol{\omega}_g &= \mathbf{D}\left((\mathbf{I} - \mathbf{W} + \mathbf{D})^\top\right)^{-1} \mathbf{e} \\
\Longrightarrow \boldsymbol{\omega}_g &= \mathbf{D}(\mathbf{I}^\top - \mathbf{W}^\top + \mathbf{D}^\top)^{-1}\mathbf{e} \\
\Longrightarrow \boldsymbol{\omega}_g &= \mathbf{D}(\mathbf{I} - \mathbf{W}^\top + \mathbf{D})^{-1}\mathbf{e} \\
\Longrightarrow (\mathbf{I} - \mathbf{W}^\top &+ \mathbf{D})\mathbf{D}^{-1}\boldsymbol{\omega}_g = \mathbf{e}
\end{aligned}$$

Let $\mathbf{A} = (\mathbf{I} - \mathbf{W}^\top + \mathbf{D})\mathbf{D}^{-1}$, so we have the linear system of $\mathbf{A}\boldsymbol{\omega}_g = \mathbf{e}$ with the solution of $\boldsymbol{\omega}_g$. For uniqueness of $\boldsymbol{\omega}_g$, we need to show that $(\mathbf{I} - \mathbf{W}^\top + \mathbf{D})\mathbf{D}^{-1}$

is a nonsingular matrix. Based on positive self-loop assumption, $\mathbf{D}$ and its inverse $\mathbf{D}^{-1}$ must be nonsingular. In the proof of Proposition 3.1, we showed that $(\mathbf{I}-\mathbf{W}+\mathbf{D})$ is nonsingular. As the transpose of any nonsingular matrix is nonsingular, $(\mathbf{I}-\mathbf{W}+\mathbf{D})^\top = (\mathbf{I}-\mathbf{W}^\top+\mathbf{D})$ is nonsingular. ∎

Lemma B3. *Assuming* $\mathbf{B} = \mathbf{W} - \mathbf{D}$ *and* $\mathbf{G} = \mathbf{D}(\mathbf{W}^\top - \mathbf{D})\mathbf{D}^{-1}$*, then* $\rho(\mathbf{G}) = \rho(\mathbf{B})$

Proof of Lemma B3 By showing that both $\mathbf{G}$ and $\mathbf{B}$ have the same characteristic polynomial (i.e., $p_B(\lambda) = p_G(\lambda)$), we demonstrate that $\sigma(\mathbf{G}) = \sigma(\mathbf{B})$, thus yielding to $\rho(\mathbf{B}) = \rho(\mathbf{G})$ based on the definition. We first note that $\mathbf{G} = \mathbf{D}\mathbf{B}^\top\mathbf{D}^{-1}$ and $det(\mathbf{D}) \neq 0$ (due to positive self-loop assumption). Then, using the definition of $p_B(\lambda)$ and transpose, multiplication and inverse properties of determinants, we have:

$$\begin{aligned}
p_B(\lambda) =& det(\mathbf{B}-\lambda\mathbf{I}) = det\left((\mathbf{B}-\lambda\mathbf{I})^\top\right) = det(\mathbf{B}^\top-\lambda\mathbf{I})\\
=&\frac{det(\mathbf{D})}{det(\mathbf{D})}det(\mathbf{B}^\top-\lambda\mathbf{I}) = det(\mathbf{D})det(\mathbf{B}^\top-\lambda\mathbf{I})det(\mathbf{D}^{-1})\\
=&det(\mathbf{D}(\mathbf{B}^\top-\lambda\mathbf{I})\mathbf{D}^{-1}) = det(\mathbf{D}\mathbf{B}^\top\mathbf{D}^{-1}-\mathbf{D}\lambda\mathbf{D}^{-1})\\
=&det(\mathbf{G}-\lambda\mathbf{I}) = p_G(\lambda)
\end{aligned}$$

Thus, $\sigma(\mathbf{G}) = \sigma(\mathbf{B})$ and consequently $\rho(\mathbf{G}) = \rho(\mathbf{B})$. ∎

Theorem 4.6. *Consider the following update:*

$$\boldsymbol{\omega}^{(t+1)} = \mathbf{D}(\mathbf{W}^\top - \mathbf{D})\mathbf{D}^{-1}\boldsymbol{\omega}^{(t)} + \mathbf{D}\mathbf{e}$$

Assuming non-negativity, normalization, and positive self-loop, this method converges to $\boldsymbol{\omega}$*, the solution to linear system stated in Theorem 4.5.*

Proof of Theorem 4.6 From Theorem 4.5, we observe that $\boldsymbol{\omega}$ is the unique solution of the linear system of $\mathbf{A}\boldsymbol{\omega} = \mathbf{e}$ where $\mathbf{A} = (\mathbf{I}-\mathbf{W}^\top+\mathbf{D})\mathbf{D}^{-1}$. The Jacobi method (as presented in Equation A.3) for solving this linear system is

$$\boldsymbol{\omega}^{(t+1)} = \boldsymbol{\Lambda}^{-1}(\mathbf{E}+\mathbf{F})\boldsymbol{\omega}^{(t)} + \boldsymbol{\Lambda}^{-1}\mathbf{e}.$$

Since $\mathbf{A} = (\mathbf{I}-\mathbf{W}^\top+\mathbf{D})\mathbf{D}^{-1} = \mathbf{D}^{-1} - (\mathbf{W}^\top-\mathbf{D})\mathbf{D}^{-1}$, we have $\boldsymbol{\Lambda} = \mathbf{D}^{-1}$ and $\mathbf{E}+\mathbf{F} = (\mathbf{W}^\top-\mathbf{D})\mathbf{D}^{-1}$ based on the definitions, thus yielding the iteration:

$$\boldsymbol{\omega}^{(t+1)} = \mathbf{D}(\mathbf{W}^\top - \mathbf{D})\mathbf{D}^{-1}\boldsymbol{\omega}^{(t)} + \mathbf{D}\mathbf{e}.$$

From Lemma B1 and Lemma B3, we have $\rho(\mathbf{G}) < 1$ where $\mathbf{G} = \mathbf{D}(\mathbf{W}^\top - \mathbf{D})\mathbf{D}^{-1}$. Then, using Corollary A1, we have shown that $\boldsymbol{\omega}^{(t+1)} = \mathbf{D}(\mathbf{W}^\top - \mathbf{D})\mathbf{D}^{-1}\boldsymbol{\omega}^{(t)} + \mathbf{De}$ converges to $\boldsymbol{\omega}$ which is the solution to the linear system of $\mathbf{A}\boldsymbol{\omega} = \mathbf{e}$ where $\mathbf{A} = (\mathbf{I} - \mathbf{W}^\top + \mathbf{D})\mathbf{D}^{-1}$. ∎

Lemma B4. *If $\boldsymbol{\omega}$ is the solution to the linear system in Theorem 4.5 and $\boldsymbol{\omega}' = (w_{11}, w_{22}, \ldots, w_{nn})^\top$ then $\boldsymbol{\omega} \geq \boldsymbol{\omega}'$.*

Proof of Lemma B4 We first show that $\boldsymbol{\omega} \geq \mathbf{0}$. As Corollary 3.1 demonstrates that $\boldsymbol{\omega}^\top = \mathbf{e}^\top(\mathbf{I} - \mathbf{W} + \mathbf{D})^{-1}\mathbf{D}$ and $\mathbf{D}$ is a non-negative matrix (due to the non-negativity assumption). It is sufficient to show that $(\mathbf{I} - \mathbf{W} + \mathbf{D})^{-1} \geq \mathbf{0}$. We can see that $\mathbf{I} - \mathbf{W} + \mathbf{D} = \mathbf{I} - (\mathbf{W} - \mathbf{D})$ is in the form of M-matrix. As $\rho(\mathbf{W} - \mathbf{D}) < 1$ (See Lemma B1), by applying Proposition A1, we have $(\mathbf{I} - \mathbf{W} + \mathbf{D})^{-1} \geq \mathbf{0}$ and consequently $\boldsymbol{\omega} \geq \mathbf{0}$.

Theorem 4.6 implies that the $\boldsymbol{\omega}$ is the fixed-point of the iterative process of $\boldsymbol{\omega}^{(t+1)} = \mathbf{D}(\mathbf{W}^\top - \mathbf{D})\mathbf{D}^{-1}\boldsymbol{\omega}^{(t)} + \mathbf{De}$. So using this and $\mathbf{De} = \boldsymbol{\omega}'$, we have $\boldsymbol{\omega} = \mathbf{D}(\mathbf{W}^\top - \mathbf{D})\mathbf{D}^{-1}\boldsymbol{\omega} + \boldsymbol{\omega}'$. As $\mathbf{W}^\top$, $\mathbf{D}$, $\mathbf{D}^{-1}$ and $\boldsymbol{\omega}^\top$ are non-negative,

$$\begin{aligned}&\boldsymbol{\omega} \geq \mathbf{0} \implies \mathbf{D}(\mathbf{W}^\top - \mathbf{D})\mathbf{D}^{-1}\boldsymbol{\omega} \geq \mathbf{0} \implies \mathbf{D}(\mathbf{W}^\top - \mathbf{D})\mathbf{D}^{-1}\boldsymbol{\omega} + \boldsymbol{\omega}' \geq \boldsymbol{\omega}' \\ &\implies \boldsymbol{\omega} \geq \boldsymbol{\omega}'. \quad \blacksquare\end{aligned}$$

Lemma B5. *Assuming non-negativity, normalization and positive self-loop, $\boldsymbol{\omega}$ in the global model always satisfies $\mathbf{e}^\top\boldsymbol{\omega} = n$ or equivalently $\sum_i \omega_i = n$.*

Proof of Lemma B5 We first note that $\mathbf{e}^\top\mathbf{W}^\top = \mathbf{e}^\top$ as a consequence of normalization assumption and also $\mathbf{e}^\top\mathbf{I} = \mathbf{e}^\top$. Then, from Theorem 4.5, we can write

$$\begin{aligned}&(\mathbf{I} - \mathbf{W}^\top + \mathbf{D})\mathbf{D}^{-1}\boldsymbol{\omega} = \mathbf{e} \implies \mathbf{e}^\top(\mathbf{I} - \mathbf{W}^\top + \mathbf{D})\mathbf{D}^{-1}\boldsymbol{\omega} = \mathbf{e}^\top\mathbf{e} \\ &\implies (\mathbf{e}^\top\mathbf{I} - \mathbf{e}^\top\mathbf{W}^\top + \mathbf{e}^\top\mathbf{D})\mathbf{D}^{-1}\boldsymbol{\omega} = n \implies (\mathbf{e}^\top - \mathbf{e}^\top + \mathbf{e}^\top\mathbf{D})\mathbf{D}^{-1}\boldsymbol{\omega} = n \\ &\implies \mathbf{e}^\top\mathbf{D}\mathbf{D}^{-1}\boldsymbol{\omega} = n \implies \mathbf{e}^\top\mathbf{I}\boldsymbol{\omega} = n \implies \mathbf{e}^\top\boldsymbol{\omega} = n \quad \blacksquare\end{aligned}$$

Theorem 4.7. *Assume $\boldsymbol{\omega}^{(0)} = (w_{11}, w_{22}, \ldots, w_{nn})^\top$. In the iterative scheme above,*

$$\left\|\boldsymbol{\omega} - \boldsymbol{\omega}^{(t)}\right\|_1 \leq n\frac{\hat{w}}{\tilde{w}}\left(1 - \tilde{w}\right)^t\left(1 - \bar{w}\right),$$

where $\tilde{w} = \min_{1 \leq j \leq n} w_{jj}$, $\hat{w} = \max_{1 \leq j \leq n} w_{jj}$, and $\bar{w} = \frac{1}{n}\sum_j w_{jj}$.

Proof of Theorem 4.7 From $\boldsymbol{\omega}^{(t)} = \mathbf{D}(\mathbf{W}^\top - \mathbf{D})\mathbf{D}^{-1}\boldsymbol{\omega}^{(t-1)} + \mathbf{De}$, we can write $\boldsymbol{\omega} - \boldsymbol{\omega}^{(t)} = \mathbf{D}(\mathbf{W}^\top - \mathbf{D})\mathbf{D}^{-1}(\boldsymbol{\omega} - \boldsymbol{\omega}^{(t-1)})$. By induction on t, we can show that $\boldsymbol{\omega} - \boldsymbol{\omega}^{(t)} = \mathbf{D}\left(\mathbf{W}^\top - \mathbf{D}\right)^t \mathbf{D}^{-1}(\boldsymbol{\omega} - \boldsymbol{\omega}^{(0)})$. Using this, we can write

$$
\begin{aligned}
\left\|\boldsymbol{\omega} - \boldsymbol{\omega}^{(t)}\right\|_1 &= \left\|\mathbf{D}\left(\mathbf{W}^\top - \mathbf{D}\right)^t \mathbf{D}^{-1}(\boldsymbol{\omega} - \boldsymbol{\omega}^{(0)})\right\|_1 \\
&\leq \|\mathbf{D}\|_1 \left\|\left(\mathbf{W}^\top - \mathbf{D}\right)^t\right\|_1 \left\|\mathbf{D}^{-1}\right\|_1 \left\|\boldsymbol{\omega} - \boldsymbol{\omega}^{(0)}\right\|_1 \\
&\leq \|\mathbf{D}\|_1 \left\|\mathbf{W}^\top - \mathbf{D}\right\|_1^t \left\|\mathbf{D}^{-1}\right\|_1 \left\|\boldsymbol{\omega} - \boldsymbol{\omega}^{(0)}\right\|_1 \\
&= \left(\max_j \sum_i |d_{ij}|\right)\left(\max_j \sum_i |w_{ji} - d_{ij}|\right)^t \left(\max_j \sum_i \left|\frac{1}{d_{ij}}\right|\right) \left\|\boldsymbol{\omega} - \boldsymbol{\omega}^{(0)}\right\|_1 \\
&= \left(\max_j |w_{jj}|\right)\left(\max_j \sum_{i \neq j} |w_{ji}|\right)^t \left(\max_j \left|\frac{1}{w_{jj}}\right|\right) \left\|\boldsymbol{\omega} - \boldsymbol{\omega}^{(0)}\right\|_1 \\
&= \left(\max_j w_{jj}\right)\left(\max_j \sum_{i \neq j} w_{ji}\right)^t \left(\frac{1}{\min_j w_{jj}}\right) \left\|\boldsymbol{\omega} - \boldsymbol{\omega}^{(0)}\right\|_1 \\
&= \left(\max_j w_{jj}\right)\left(1 - \min_j w_{jj}\right)^t \left(\frac{1}{\min_j w_{jj}}\right) \left\|\boldsymbol{\omega} - \boldsymbol{\omega}^{(0)}\right\|_1 \\
&= \left(\frac{\max_j w_{jj}}{\min_j w_{jj}}\right)\left(1 - \min_j w_{jj}\right)^t \left(\sum_j |\omega_j - w_{jj}|\right) \\
&= \left(\frac{\max_j w_{jj}}{\min_j w_{jj}}\right)\left(1 - \min_j w_{jj}\right)^t \left(\sum_j (\omega_j - w_{jj})\right) \quad \text{Lemma B4} \\
&= \left(\frac{\max_j w_{jj}}{\min_j w_{jj}}\right)\left(1 - \min_j w_{jj}\right)^t \left(n - \sum_j w_{jj}\right) \quad \text{Lemma B5} \\
&= n\left(\frac{\max_j w_{jj}}{\min_j w_{jj}}\right)\left(1 - \min_j w_{jj}\right)^t \left(1 - \frac{1}{n}\sum_j w_{jj}\right)
\end{aligned}
$$

Let $\tilde{w} = \min_{1 \leq j \leq n} w_{jj}$, $\hat{w} = \max_{1 \leq j \leq n} w_{jj}$, and $\bar{w} = \frac{1}{n}\sum_j w_{jj}$. Thus,

$$
\left\|\boldsymbol{\omega} - \boldsymbol{\omega}^{(t)}\right\|_1 \leq n\frac{\hat{w}}{\tilde{w}}\left(1 - \tilde{w}\right)^t (1 - \bar{w}).
$$

■

Theorem 4.8. *Assume* $\boldsymbol{\omega}^{(0)} = (w_{11}, w_{22}, \ldots, w_{nn})^\top$*. Under normalization, non-negativity, and positive self-loop, for any* t*:*

$$|sw(a) - sw^{(t)}(a)| \le n\frac{\hat{w}}{\tilde{w}}(1-\tilde{w})^t(1-\bar{w})\left\|\mathbf{u}^I(a)\right\|_2,$$

where $\tilde{w} = \min_{1\le j\le n} w_{jj}$*,* $\hat{w} = \max_{1\le j\le n} w_{jj}$*, and* $\bar{w} = \frac{1}{n}\sum_j w_{jj}$*.*

Proof of Theorem 4.8 For $sw^{(t)}(a)$ and $sw(a)$, using the Cauchy-Schwarz inequality, we can write:

$$\begin{aligned}|sw(a) - sw^{(t)}(a)| &= |\boldsymbol{\omega}^\top \mathbf{u}^I(a) - (\boldsymbol{\omega}^{(t)})^\top \mathbf{u}^I(a)| \\ &= |(\boldsymbol{\omega} - \boldsymbol{\omega}^{(t)})^\top \mathbf{u}^I(a)| \le \left\|\boldsymbol{\omega} - \boldsymbol{\omega}^{(t)}\right\|_2 \left\|\mathbf{u}^I(a)\right\|_2\end{aligned}$$

In general, for a given vector $\mathbf{x}$, $\|\mathbf{x}\|_2 \le \|\mathbf{x}\|_1$. Thus, we here have $|sw(a) - sw^{(t)}(a)| \le \left\|\boldsymbol{\omega} - \boldsymbol{\omega}^{(t)}\right\|_1 \left\|\mathbf{u}^I(a)\right\|_2$. By applying Theorem 4.7, we have:

$$|sw(a) - sw^{(t)}(a)| \le \left\|\boldsymbol{\omega} - \boldsymbol{\omega}^{(t)}\right\|_1 \left\|\mathbf{u}^I(a)\right\|_2 \le n\frac{\hat{w}}{\tilde{w}}(1-\tilde{w})^t(1-\bar{w})\left\|\mathbf{u}^I(a)\right\|_2$$

where $\tilde{w} = \min_{1\le j\le n} w_{jj}$, $\hat{w} = \max_{1\le j\le n} w_{jj}$, and $\bar{w} = \frac{1}{n}\sum_j w_{jj}$. Thus, we have shown

$$|sw(a) - sw^{(t)}(a)| \le n\frac{\hat{w}}{\tilde{w}}(1-\tilde{w})^t(1-\bar{w})\left\|\mathbf{u}^I(a)\right\|_2. \quad \blacksquare$$

Proposition 4.9. *If* $sw^{(t)}(a) - sw^{(t)}(b) \ge n\frac{\hat{w}}{\tilde{w}}(1-\tilde{w})^t(1-\bar{w})\left(\left\|\mathbf{u}^I(a)\right\|_2 + \left\|\mathbf{u}^I(b)\right\|_2\right)$ *then* $sw(a) > sw(b)$*.*

Proof of Proposition 4.9 Using the inequality presented in Theorem 4.8, we can write:

$$\begin{aligned}sw^{(t)}(a) - sw^{(t)}(b) &= sw^{(t)}(a) - sw(a) + sw(a) - sw^{(t)}(b) + sw(b) - sw(b) \\ &\le |sw^{(t)}(a) - sw(a)| + sw(a) + |sw(b) - sw^{(t)}(b)| - sw(b) \\ &\le n\frac{\hat{w}}{\tilde{w}}(1-\tilde{w})^t(1-\bar{w})\left\|\mathbf{u}^I(a)\right\|_2 + sw(a) \\ &\quad + n\frac{\hat{w}}{\tilde{w}}(1-\tilde{w})^t(1-\bar{w})\left\|\mathbf{u}^I(b)\right\|_2 - sw(b) \\ &= n\frac{\hat{w}}{\tilde{w}}(1-\tilde{w})^t(1-\bar{w})\left(\left\|\mathbf{u}^I(a)\right\|_2 + \left\|\mathbf{u}^I(b)\right\|_2\right) + sw(a) - sw(b)\end{aligned}$$

Using this and $sw^{(t)}(a) - sw^{(t)}(b) \ge n\frac{\hat{w}}{\tilde{w}}(1-\tilde{w})^t(1-\bar{w})\left(\left\|\mathbf{u}^I(a)\right\|_2 + \left\|\mathbf{u}^I(b)\right\|_2\right)$, we have $sw(a) \ge sw(b)$. $\blacksquare$